Various structures of T-definable functionals via a Gentzen-style translation

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Introduction and motivations

This talk is to

1. to present a monadic translation of Gödel's System T into itself which is in the spirit of the Gentzen's negative translation of logic, and
2. to demonstrate how various structures of T-definable functions can be directly revealed via its instantiations.

Motivations:

- [Oliva & Steila 2018]: bar recursion closure theorem
- [Escardó 2013]: dialogue trees
- [van den Berg 2019]: generalization of Kuroda's negative translation

Gödel's system T

We work with (the term language of) Gödel's System T in its λ-calculus form

$$\mathrm{T} \;\equiv\; \text{simply typed } \lambda\text{-calculus} \;+\; \mathbb{N} \;+\; \text{primitive recursor.}$$

We extend T with products and sums. Hence, T can be given by

$$\begin{array}{ll} \text{Type} & \sigma, \tau :\equiv \mathbb{N} \mid \sigma \to \tau \mid \sigma \times \tau \mid \sigma + \tau \\ \text{Term} & t, u :\equiv x \mid \lambda x.t \mid tu \mid \mathsf{c} \end{array}$$

where constants c include those for

- natural numbers:

$$0 : \mathbb{N} \qquad \mathrm{suc} : \mathbb{N} \to \mathbb{N} \qquad \mathrm{rec} : \sigma \to (\mathbb{N} \to \sigma \to \sigma) \to \mathbb{N} \to \sigma$$

- products:

$$\mathrm{pair} : \sigma_1 \to \sigma_2 \to \sigma_1 \times \sigma_2 \qquad \mathrm{pr}_i : \sigma_1 \times \sigma_2 \to \sigma_i$$

- sums:

$$\mathrm{inj}_i : \sigma_i \to \sigma_1 + \sigma_2 \qquad \mathrm{case} : (\sigma_1 \to \tau) \to (\sigma_2 \to \tau) \to \sigma_1 + \sigma_2 \to \tau$$

Gödel's system T: some conventions

A function is called T-definable if we can find a term in T denoting it.

But in this talk, we do not distinguish T-definable functions and their corresponding terms in T.

Moreover, we (may) write

- $\lambda x_1 x_2 \cdots x_n.t$ instead of $\lambda x_1.\lambda x_2.\cdots\lambda x_n.t$;
- $f(a_1, a_2, \cdots, a_n)$ instead of $(((f a_1) a_2) \cdots) a_n$;
- $\langle a, b\rangle$ instead of $\mathrm{pair}(a, b)$;
- w_i instead of $\mathrm{pr}_i w$ for $i \in \{1, 2\}$;
- $n + 1$ instead of $\mathrm{suc}(n)$;
- $\mathbb{N}^{\mathbb{N}}$ instead of $\mathbb{N} \to \mathbb{N}$;
- α_i instead of $\alpha(i)$ for $\alpha : \mathbb{N}^{\mathbb{N}}$ and $i : \mathbb{N}$.

Gentzen's negative translation and its generalization

Translating formulas in predicate logic as follows

$$(A \to B)^{\mathrm{G}} :\equiv A^{\mathrm{G}} \to B^{\mathrm{G}} \qquad P^{\mathrm{G}} :\equiv \neg\neg P \quad \text{for primitive } P$$
$$(A \wedge B)^{\mathrm{G}} :\equiv A^{\mathrm{G}} \wedge B^{\mathrm{G}} \qquad (A \vee B)^{\mathrm{G}} :\equiv \neg\neg (A^{\mathrm{G}} \vee B^{\mathrm{G}})$$
$$(\forall x A)^{\mathrm{G}} :\equiv \forall x A^{\mathrm{G}} \qquad (\exists x A)^{\mathrm{G}} :\equiv \neg\neg \exists x A^{\mathrm{G}}$$

one can prove $\mathrm{CL} \vdash A \iff \mathrm{ML} \vdash A^{\mathrm{G}}$.

This translation can be generalized by replacing $\neg\neg$ by arbitrary nuclei[1,2], that is, a mapping j on formulas satisfying certain conditions.

- For any j, we have $\mathrm{IL} \vdash A \Longrightarrow \mathrm{IL} \vdash A^{\mathrm{G}}_j$.
- If $jA = (A \to R) \to R$, then $\mathrm{CL} \vdash A \Longrightarrow \mathrm{IL} \vdash A^{\mathrm{G}}_j$.
- If $jA = A \vee \bot$, then $\mathrm{IL} \vdash A \Longrightarrow \mathrm{ML} \vdash A^{\mathrm{G}}_j$.

[1] M. Escardó and P. Oliva. *The Peirce translation*, Annals of Pure and Applied Logic 163 (6), pp. 681–692, 2012.

[2] B. van den Berg, *A Kuroda-style j-translation*, Archive for Mathematical Logic 58 (5–6), pp. 627–634, 2019.

Nuclei (relative to T)

A nucleus (relative to T) is an endofunction J on types of T equipped with T-terms

$$\eta : \rho \to \mathrm{J}\rho \qquad \kappa : (\sigma \to \mathrm{J}\rho) \to \mathrm{J}\sigma \to \mathrm{J}\rho$$

for any types σ, ρ such that

$$\eta^\kappa = \mathrm{id} \qquad f^\kappa \circ \eta = f \qquad (g^\kappa \circ f)^\kappa = g^\kappa \circ f^\kappa$$

hold up to pointwise equality, where we write f^κ to denote κf.

For any nucleus J, we can define the following terms in T:

- $\mu :\equiv (\lambda x^{\mathrm{J}\rho}.x)^\kappa : \mathrm{J}\mathrm{J}\rho \to \mathrm{J}\rho$
- $J :\equiv \lambda f^{\sigma \to \rho}.(\eta \circ f)^\kappa : (\sigma \to \rho) \to \mathrm{J}\sigma \to \mathrm{J}\rho$

Hence (J, μ, η) forms a monad on the term model of T.

A Gentzen-style translation of T

We translate types of T in the style of Gentzen

$$(\sigma \to \tau)^{\mathrm{J}} :\equiv \sigma^{\mathrm{J}} \to \tau^{\mathrm{J}} \qquad \mathbb{N}^{\mathrm{J}} :\equiv \mathrm{J}\mathbb{N}$$

$$(\sigma \times \tau)^{\mathrm{J}} :\equiv \sigma^{\mathrm{J}} \times \tau^{\mathrm{J}} \qquad (\sigma + \tau)^{\mathrm{J}} :\equiv \mathrm{J}(\sigma^{\mathrm{J}} + \tau^{\mathrm{J}})$$

Assume a mapping of variables $x : \sigma$ to $x^{\mathrm{J}} : \sigma^{\mathrm{J}}$. For each term $t : \rho$ of T, we assign a term $t^{\mathrm{J}} : \rho^{\mathrm{J}}$ by

$$(x)^{\mathrm{J}} :\equiv x^{\mathrm{J}} \qquad 0^{\mathrm{J}} :\equiv \eta(0)$$

$$(\lambda x.t)^{\mathrm{J}} :\equiv \lambda x^{\mathrm{J}}.t^{\mathrm{J}} \qquad \mathrm{suc}^{\mathrm{J}} :\equiv J(\mathrm{suc})$$

$$(tu)^{\mathrm{J}} :\equiv t^{\mathrm{J}} u^{\mathrm{J}} \qquad \mathrm{rec}^{\mathrm{J}} :\equiv \lambda a f.\mathrm{ke}(\mathrm{rec}(a, f \circ \eta))$$

$$\mathrm{pair}^{\mathrm{J}} :\equiv \mathrm{pair} \qquad \mathrm{inj}_i^{\mathrm{J}} :\equiv \eta \circ \mathrm{inj}_i$$

$$\mathrm{pr}_i^{\mathrm{J}} :\equiv \mathrm{pr}_i \qquad \mathrm{case}^{\mathrm{J}} :\equiv \lambda f g.\mathrm{ke}(\mathrm{case}(f, g))$$

corresponding to the soundness proof of Gentzen's negative translation.

Kleisli extension

Given $a : \rho^{\mathrm{J}}$ and $f : \mathrm{J}\mathbb{N} \to \rho^{J} \to \rho^{\mathrm{J}}$, we want to define $\mathrm{rec}^{\mathrm{J}}(a, f) : \mathrm{J}\mathbb{N} \to \rho^{\mathrm{J}}$.

A promising candidate is $\mathrm{rec}(a, f \circ \eta) : \mathbb{N} \to \rho^{\mathrm{J}}$.

But we cannot directly use $\kappa : (\sigma \to \mathrm{J}\rho) \to \mathrm{J}\sigma \to \mathrm{J}\rho$.

We define $\mathrm{ke}^{\sigma}_{\rho} : (\sigma \to \rho^{\mathrm{J}}) \to \mathrm{J}\sigma \to \rho^{\mathrm{J}}$ by induction on ρ as follows

$$\begin{aligned} \mathrm{ke}^{\sigma}_{\mathbb{N}}(f, a) &:\equiv f^{\kappa} a \\ \mathrm{ke}^{\sigma}_{\tau+\rho}(f, a) &:\equiv f^{\kappa} a \\ \mathrm{ke}^{\sigma}_{\tau\to\rho}(f, a) &:\equiv \lambda x^{\tau^{\mathrm{J}}}.\mathrm{ke}^{\sigma}_{\rho}(\lambda y^{\sigma}.f(y, x), a) \\ \mathrm{ke}^{\sigma}_{\tau\times\rho}(f, a) &:\equiv \langle \mathrm{ke}^{\sigma}_{\tau}(\mathrm{pr}_1 \circ f, a), \mathrm{ke}^{\sigma}_{\rho}(\mathrm{pr}_2 \circ f, a)\rangle. \end{aligned}$$

and then use it to define $\mathrm{rec}^{\mathrm{J}}$ and $\mathrm{case}^{\mathrm{J}}$.

Lemma (Kleisli extension). For any $f : \sigma \to \rho^{\mathrm{J}}$ and $x : \sigma$, we have

$$\mathrm{ke}^{\sigma}_{\rho}(f, \eta x) = f x.$$

Correctness

Lemma. The J-translation preserves substitutions, *i.e.*

$$(t[u/x])^{\mathrm{J}} = t^{\mathrm{J}}[u^{\mathrm{J}}/x^{\mathrm{J}}].$$

Theorem (Correctness).

- If $\Gamma \vdash t : \rho$, then $\Gamma^{\mathrm{J}} \vdash t^{\mathrm{J}} : \rho^{\mathrm{J}}$.
- If $t =_{\beta\eta} u$, then $t^{\mathrm{J}} =_{\beta\eta} u^{\mathrm{J}}$.

The examples in this talk use only the Kleisli-extension lemma.

For simplicity, we consider T without sums in the examples.

Any nucleus on natural numbers (*i.e.* a type $\mathrm{J}\mathbb{N}$ with terms $\eta : \mathbb{N} \to \mathrm{J}\mathbb{N}$ and $\kappa : (\mathbb{N} \to \mathrm{J}\mathbb{N}) \to \mathrm{J}\mathbb{N} \to \mathrm{J}\mathbb{N}$) suffices to translate T without sums.

Example I: lifting to functions of higher type levels[3]

If one wants to prove a property P of functions $f : X \to \mathbb{N}$ (such as continuity of functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$), the usual syntactic method using an inductively defined logical relation may not work directly.

We "precook" T by applying the J-translation with the following nucleus

$$\mathrm{J}\mathbb{N} :\equiv X \to \mathbb{N} \qquad \eta(n) :\equiv \lambda x.n \qquad f^{\kappa}(g) :\equiv \lambda x.f(gx, x).$$

For any concrete type X, we can construct a term $\Omega : X^{\mathrm{J}}$ such that

$$f^{\mathrm{J}}(\Omega) = f$$

up to pointwise equality, for any $f : X \to \mathbb{N}$ of T.

[3] C. Xu, *A syntactic approach to continuity of T-definable functionals*, `arXiv:1904.09794 [math.LO]` (2019).

Example I: lifting to functions of higher type levels (cont.)

Define a predicate $Q_\rho \subseteq \rho^{\mathrm{J}}$ inductively on ρ

$$Q_{\mathbb{N}}(f) :\equiv P(f) \qquad \text{the desired property}$$

$$Q_{\sigma\to\tau}(h) :\equiv \forall x^{\sigma^{\mathrm{J}}} \left(Q_\sigma(x) \to Q_\tau(hx)\right).$$

Once we prove (1) $Q_\rho(t^{\mathrm{J}})$ for all $t : \rho$ of T and (2) $Q_X(\Omega)$, we can conclude

$$P(f) \text{ for all } f : X \to \mathbb{N} \text{ in } \mathrm{T}$$

because we have $Q_{\mathbb{N}}(f^{\mathrm{J}}\Omega) = P(f^{\mathrm{J}}\Omega)$ and $f = f^{\mathrm{J}}\Omega$.

All the examples presented later can be proved using this method.

But we can instead work with a nucleus J which reflects the computational information of the property P, so that witnesses of P can be obtained as terms of T directly via the J-translation.

Example II: majorizability[4]

Recall that the relation $\mathrm{maj}_\rho \subseteq \rho \times \rho$ is defined by

$$n \ \mathrm{maj}_{\mathbb{N}} \ m \ :\equiv \ n \geq m$$
$$f \ \mathrm{maj}_{\sigma\to\tau} \ g \ :\equiv \ \forall x^\sigma y^\sigma \left(x \ \mathrm{maj}_\sigma \ y \to fx \ \mathrm{maj}_\tau \ gy \right).$$

Consider the nucleus

$$\mathrm{J}\mathbb{N} :\equiv \mathbb{N} \qquad \eta(n) :\equiv n \qquad \begin{cases} g^\kappa(0) :\equiv g(0) \\ g^\kappa(n+1) :\equiv \max(g^\kappa(n), g(n+1)). \end{cases}$$

Theorem. For any $t : \rho$ of T, we have

$$t^{\mathrm{J}} \ \mathrm{maj}_\rho \ t.$$

[4] W. A. Howard. *Hereditarily majorizable functionals of finite type*. In Metamathematical investigation of intuitionistic Arithmetic and Analysis, volume 344 of Lecture Notes in Mathematics, pages 454–461. Springer, Berlin, 1973.

Example III: continuity[5]

Recall that $M : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is called a modulus of continuity of $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ if

$$\forall \alpha\, \beta\, (\alpha =_{M\alpha} \beta \to f\alpha = f\beta)\,.$$

Consider the nucleus

$$\mathrm{J}\mathbb{N} :\equiv \mathbb{N} \times \mathbb{N} \qquad \eta(n) :\equiv \langle n, 0 \rangle \qquad g^{\kappa}(x) :\equiv \langle (g x_1)_1, \max(x_2, (g x_1)_2) \rangle.$$

Given $\alpha : \mathbb{N}^{\mathbb{N}}$, we construct a term $\tilde{\alpha} : \mathrm{J}\mathbb{N} \to \mathrm{J}\mathbb{N}$ by

$$\tilde{\alpha} :\equiv (\lambda n. \langle \alpha_n, n+1 \rangle)^{\kappa}.$$

Theorem. For any $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ of T, the term $M : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ defined by

$$M :\equiv \lambda \alpha. (f^{\mathrm{J}} \tilde{\alpha})_2$$

is a modulus of continuity of f.

[5] M. H. Escardó, *Continuity of Gödel's system T functionals via effectful forcing*, MFPS'2013. Electronic Notes in Theoretical Computer Science 298 (2013), 119–141.

Example III: continuity – intuition

- An element of $\mathrm{J}\mathbb{N}$ $(:\equiv \mathbb{N} \times \mathbb{N})$ is a pair $\langle v, m \rangle$ where
  - v is the value of some function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ at some point $\alpha : \mathbb{N}^{\mathbb{N}}$ and
  - m is a modulus of continuity of f at α.
- $\eta(n) :\equiv \langle n, 0 \rangle$ represents the constant function with value n.
- $g^{\kappa} :\equiv \lambda x.\langle (gx_1)_1, \max(x_2, (gx_1)_2) \rangle$ is the extension of $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ with the modulus updated in a reasonable way.

Now assume that we have a function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ and an input $\alpha : \mathbb{N}^{\mathbb{N}}$.

- $f^{\mathrm{J}} : (\mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}) \to \mathbb{N} \times \mathbb{N}$ computes also a modulus.
- The generic element $\tilde{\alpha} : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$

$$\tilde{\alpha} :\equiv (\lambda n.\langle \alpha_n, n+1 \rangle)^{\kappa} = \lambda x.\langle \alpha_{x_1}, \max(x_2, x_1 + 1) \rangle$$

  updates the modulus if a larger index of α is used.
- Applying f^{J} to $\tilde{\alpha}$, we get both the value $(f^{\mathrm{J}}\tilde{\alpha})_1$ and modulus $(f^{\mathrm{J}}\tilde{\alpha})_2$.

Example III: continuity – proof

Proof. We use also the lifting nucleus $\mathrm{b}\mathbb{N} :\equiv \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ introduced in Example I and write $t^{\mathrm{b}} : \rho^{\mathrm{b}}$ to denote the translation with the nucleus b.

Given $\alpha : \mathbb{N}^{\mathbb{N}}$, define a logical relation $\mathbf{R}^{\alpha}_{\rho} \subseteq \rho^{\mathrm{J}} \times \rho^{\mathrm{b}}$ by

$$w \,\mathbf{R}^{\alpha}_{\mathbb{N}}\, f \;:\equiv\; w_1 = f\alpha \wedge \forall \beta \, (\alpha =_{w_2} \beta \to f\alpha = f\beta)$$
$$g \,\mathbf{R}^{\alpha}_{\sigma \to \tau}\, h \;:\equiv\; \forall x\, y \, (x \,\mathbf{R}^{\alpha}_{\sigma}\, y \to gx \,\mathbf{R}^{\alpha}_{\tau}\, hy)$$

We can prove for any $\alpha : \mathbb{N}^{\mathbb{N}}$

(i) $t^{\mathrm{J}} \,\mathbf{R}^{\alpha}_{\rho}\, t^{\mathrm{b}}$ for any $t : \rho$ of T, and

(ii) $\tilde{\alpha} \,\mathbf{R}^{\alpha}_{\mathbb{N}\to\mathbb{N}}\, \Omega$ where $\Omega :\equiv \lambda f \alpha . \alpha(f\alpha) : (\mathbb{N}^{\mathbb{N}} \to \mathbb{N}) \to \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$.

Then for any $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ of T, we have

- $f = f^{\mathrm{b}}\Omega$ up to pointwise equality (Example I),
- $(f^{\mathrm{J}}\tilde{\alpha})_2$ is a modulus of continuity of $f^{\mathrm{b}}\Omega$ at α.

Hence $M :\equiv \lambda\alpha.(f^{\mathrm{J}}\tilde{\alpha})_2$ is a modulus of continuity of f. □

Example IV: bar recursion[6]

Let $S : \mathbb{N}^* \to \mathbf{2}$ be a monotone function. We write $S(s)$ to denote $S(s) = 1$.

We call $\xi : (\mathbb{N}^* \to \sigma) \to (\mathbb{N}^* \to \sigma^{\mathbb{N}} \to \sigma) \to \mathbb{N}^* \to \sigma$ a functional of general bar recursion for S if $\mathcal{GBR}_S(\xi)$ holds, *i.e.*

$$\forall G^{\mathbb{N}^* \to \sigma} H^{\mathbb{N}^* \to \sigma^{\mathbb{N}} \to \sigma} s^{\mathbb{N}^*} \left\{ \begin{array}{c} S(s) \to \xi(G,H,s) = G(s) \\ \wedge \\ \neg S(s) \to \xi(G,H,s) = H(s, \lambda x.\xi(G,H,s*x)) \end{array} \right\}.$$

We say S secures $Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ if

$$\forall s^{\mathbb{N}^*} \left(S(s) \to \forall \alpha^{\mathbb{N}^{\mathbb{N}}} Y(s * 0^{\omega}) = Y(s * \alpha) \right).$$

Theorem [Oliva&Steila2018]. If S secures Y, then from any functional ξ of general bar recursion for S one can construct a functional $\Phi^Y(\xi)$ of Spector's bar recursion for Y.

[6] P. Oliva, S. Steila, *A direct proof of Schwichtenberg's bar recursion closure theorem*, The Journal of Symbolic Logic 83 (1) (2018) 70–83.

Example IV: bar recursion – construction

We extend T with ρ^* and $\mathbb{2}$. Fix a type σ. Let

$$\mathrm{J}\mathbb{N} :\equiv (\mathbb{N}^{\mathbb{N}} \to \mathbb{N}) \times (\mathbb{N}^* \to \mathbb{2}) \times ((\mathbb{N}^* \to \sigma) \to (\mathbb{N}^* \to \sigma^{\mathbb{N}} \to \sigma) \to \mathbb{N}^* \to \sigma)$$

and write $\mathrm{V}_x, \mathrm{S}_x, \mathrm{B}_x$ to denote the three components of $x : \mathrm{J}\mathbb{N}$. Define

$$\begin{aligned} \eta(n) &:\equiv \langle \lambda\alpha.n,\ \lambda s.1,\ \lambda GH.G \rangle \\ g^\kappa(x) &:\equiv \langle \lambda\alpha.\mathrm{V}_{g(\mathrm{V}_x\alpha)}\alpha, \\ &\qquad \lambda s.\min(\mathrm{S}_x(s), \mathrm{S}_{g(\mathrm{V}_x(s*0^\omega))}(s)), \\ &\qquad \lambda GH.\mathrm{B}_x(\lambda s.\mathrm{B}_{g(\mathrm{V}_x(s*0^\omega))}(G,H,s),H) \rangle. \end{aligned}$$

We construct the generic element $\Omega : \mathrm{J}\mathbb{N} \to \mathrm{J}\mathbb{N}$ by

$$\Omega :\equiv (\lambda n.\langle \lambda\alpha.\alpha n, \lambda s.\mathrm{Le}(n,|s|), \Psi n \rangle)^\kappa$$

where $\mathrm{Le} : \mathbb{N} \to \mathbb{N} \to \mathbb{2}$ has value 1 iff the first argument is smaller, and $\Psi n : (\mathbb{N}^* \to \sigma) \to (\mathbb{N}^* \to \sigma^{\mathbb{N}} \to \sigma) \to \mathbb{N}^* \to \sigma$ is a (T-definable) functional of bar recursion for constant Y with value n ([Oliva&Steila2018, Lemma 2.1]).

Example IV: bar recursion – correctness

Theorem. For any $Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ of T,

- $\mathrm{S}_{Y^{\mathrm{J}}\Omega}$ is a monotone function securing Y, and
- $\mathrm{B}_{Y^{\mathrm{J}}\Omega}$ is a functional of general bar recursion for $\mathrm{S}_{Y^{\mathrm{J}}\Omega}$.

Proof. Work with the logical relation $\mathbf{R}^{\alpha}_{\rho} \subseteq \rho^{\mathrm{J}} \times \rho$ parametrized by $\alpha : \mathbb{N}^{\mathbb{N}}$

$$w \, \mathbf{R}^{\alpha}_{\mathbb{N}} \, n \; :\equiv \; \mathrm{V}_w \alpha = n \wedge \mathrm{S}_w \text{ secures } \mathrm{V}_w \wedge \mathcal{GBR}_{\mathrm{S}_w}(\mathrm{B}_w)$$
$$g \, \mathbf{R}^{\alpha}_{\sigma \to \tau} \, h \; :\equiv \; \forall x \, y \, (x \, \mathbf{R}^{\alpha}_{\sigma} \, y \to gx \, \mathbf{R}^{\alpha}_{\tau} \, hy) \, .$$

Prove (i) $t^{\mathrm{J}} \, \mathbf{R}^{\alpha}_{\rho} \, t$ for all $t : \rho$ of T, and (ii) $\Omega \, \mathbf{R}^{\alpha}_{\mathbb{N} \to \mathbb{N}} \, \alpha$, which together bring the desired result. □

Corollary. For any $Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ of T, the term

$$\Phi^{Y}(\mathrm{B}_{Y^{\mathrm{J}}\Omega})$$

is a functional of Spector's bar recursion for Y.

Other variants of monadic translation

In the Kolmogorov-style[7], type are translated by $\sigma^{\mathrm{Ko}} :\equiv \mathrm{J}\langle\sigma\rangle$ where

$$\langle\mathbb{N}\rangle :\equiv \mathbb{N}$$

$$\langle\sigma \mathbin{\square} \tau\rangle :\equiv \mathrm{J}\langle\sigma\rangle \mathbin{\square} \mathrm{J}\langle\tau\rangle \quad \text{for } \square \in \{\to, \times, +\}$$

In the Kuroda-style[8,9], types are translated by $\sigma^{\mathrm{Ku}} :\equiv \mathrm{J}[\sigma]$ where

$$[\mathbb{N}] :\equiv \mathbb{N} \qquad [\sigma \times \tau] :\equiv [\sigma] \times [\tau]$$

$$[\sigma \to \tau] :\equiv [\sigma] \to \mathrm{J}[\tau] \qquad [\sigma + \tau] :\equiv [\sigma] + [\tau]$$

Both require nonstandard notions of application when translating function application.

[7] T. Uustalu, *Monad translating inductive and coinductive types*, in: Types for Proofs and Programs (TYPES 2002). Lecture Notes in Computer Science, vol 2646, Springer, 2002, pp. 299–315.

[8] T. Coquand and G. Jaber. *A computational interpretation of forcing in type theory*. Epistemology versus Ontology, volume 27, pages 203–213. Springer Netherlands, 2012.

[9] T. Powell, *A functional interpretation with state*, in: Proceedings of the Thirty third Annual IEEE Symposium on Logic in Computer Science (LICS 2018), IEEE Computer Society Press, 2018, pp. 839–848.

Summary

- We introduce a syntactic translation of T, parametrized by a notion of nucleus relative to T, in the style of Gentzen.
- Working with different nuclei, we construct
  - majorants,
  - moduli of (uniform) continuity, and
  - general bar recursion functionals

  of T-definable functions directly via the translation.
- Preprint: `arXiv:1908.05979 [cs.LO]`
- All the above work has been implemented in Agda, which is available at

  http://cj-xu.github.io/agda/ModTrans/index.html

Thank you! Questions and comments are very appreciated!