A syntactic approach to continuity of Gödel's system T definable functionals

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This talk is

1. to present a syntactic approach to continuity of functions $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ that are definable in Gödel's system T, and

2. to generalize the method to prove other properties/structures of T-definable functionals.

Gödel's system T

We work with (the term language of) Gödel's system T in its λ-calculus form

$$\mathrm{T} \;\equiv\; \text{simply typed } \lambda\text{-calculus} \;+\; \mathbb{N} \;+\; \text{primitive recursor.}$$

Finite types defined inductively

$$\rho, \sigma, \tau \;\;:\equiv\;\; \mathbb{N} \mid \sigma \to \tau$$

Constants associated to the ground type $\mathbb{N}$ include

- $0 : \mathbb{N}$,
- $\mathrm{succ} : \mathbb{N} \to \mathbb{N}$, and
- $\mathrm{rec} : \rho \to (\mathbb{N} \to \rho \to \rho) \to \mathbb{N} \to \rho$ with

$$\mathrm{rec}(a)(f)(0) = a \qquad \mathrm{rec}(a)(f)(\mathrm{succ}(n)) = f(n)(\mathrm{rec}(a)(f)(n))$$

for every finite type ρ.

Syntactic method

Usual steps:

(1) Define a predicate $P_\rho \subseteq \rho$ by induction on the finite type ρ

$$\begin{array}{rcl} P_{\mathbb{N}}(n) & :\equiv & \cdots \\ P_{\sigma\to\tau}(h) & :\equiv & \forall x^\sigma \, (P_\sigma(x) \to P_\tau(hx)) \end{array}$$

(2) Prove that any term $t : \rho$ in T satisfies P_ρ by induction on the term t.

Examples: totality[1], majorizability[2], ...

However, what we want to prove is continuity of functions $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ which is a $\forall\exists\forall$-statement. It seems impossible to define a $P_{\mathbb{N}}$ such that $P_{(\mathbb{N}\to\mathbb{N})\to\mathbb{N}}(f)$ expresses the continuity of f.

[1] H. Schwichtenberg and S. S. Wainer, *Proofs and Computations*. Cambridge University Press, 2012.

[2] U. Kohlenbach, *Applied Proof Theory: Proof Interpretations and Their Use in Mathematics*. Springer-Verlag Berlin Heidelberg, 2008.

Precook T – b-translation

The idea is, as a Step (0), to perform a translation $(t \mapsto t^{\mathrm{b}}) : \rho \to \rho^{\mathrm{b}}$ from T to itself such that continuity of functions $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ becomes the base case of the predicate $\mathrm{C}_\rho \subseteq \rho^{\mathrm{b}}$ defined in (1). Once we know that the translation f^{b} satisfies the predicate which is a result of (2), then f is continuous.

- For each finite type ρ we associate inductively a new one ρ^{b} as

$$\begin{aligned} \mathbb{N}^{\mathrm{b}} &:\equiv (\mathbb{N} \to \mathbb{N}) \to \mathbb{N} \\ (\sigma \to \tau)^{\mathrm{b}} &:\equiv \sigma^{\mathrm{b}} \to \tau^{\mathrm{b}}. \end{aligned}$$

- For any term $t : \rho$ in T, we define $t^{\mathrm{b}} : \rho^{\mathrm{b}}$ inductively as follows

$$\begin{aligned} (x^\rho)^{\mathrm{b}} &:\equiv x^{\mathrm{b}} \\ (\lambda x.u)^{\mathrm{b}} &:\equiv \lambda x^{\mathrm{b}}.u^{\mathrm{b}} \\ (fa)^{\mathrm{b}} &:\equiv f^{\mathrm{b}} a^{\mathrm{b}} \\ 0^{\mathrm{b}} &:\equiv \lambda \alpha.0 \\ \mathrm{succ}^{\mathrm{b}} &:\equiv \lambda f.(\mathrm{succ} \circ f) \\ \mathrm{rec}^{\mathrm{b}} &:\equiv ??? \end{aligned}$$

Precook T – Kleisli extension

To have a sound translation, $\mathrm{rec}^{\mathrm{b}} : \rho^{\mathrm{b}} \to (\mathbb{N}^{\mathrm{b}} \to \rho^{\mathrm{b}} \to \rho^{\mathrm{b}}) \to \mathbb{N}^{\mathrm{b}} \to \rho^{\mathrm{b}}$ has to preserve the computational rules of rec, *i.e.* $\mathrm{rec}^{\mathrm{b}}$ should satisfy

$$\mathrm{rec}^{\mathrm{b}}(a)(f)(0^{\mathrm{b}}) = a \qquad \mathrm{rec}^{\mathrm{b}}(a)(f)(\mathrm{succ}\ n)^{\mathrm{b}} = f(n^{\mathrm{b}})(\mathrm{rec}^{\mathrm{b}}(a)(f)(n^{\mathrm{b}}))$$

A candidate for $\mathrm{rec}^{\mathrm{b}}(a)(f) : \mathbb{N}^{\mathrm{b}} \to \rho^{\mathrm{b}}$ is $\mathrm{rec}(a)(\lambda k.f(k^{\mathrm{b}})) : \mathbb{N} \to \rho^{\mathrm{b}}$.

It is possible to extend $g : \mathbb{N} \to \rho^{\mathrm{b}}$ to $g^* : \mathbb{N}^{\mathrm{b}} \to \rho^{\mathrm{b}}$ such that $\forall i.\ g^*(i^{\mathrm{b}}) = g(i)$ by induction on ρ – Kleisli extension

$$\begin{aligned} \mathrm{ke}^{\mathbb{N}}(g)(f) &:\equiv \lambda\alpha.g(f\alpha)(\alpha) \\ \mathrm{ke}^{\sigma\to\tau}(g)(f) &:\equiv \lambda x.\mathrm{ke}^{\tau}(\lambda k.g(k)(x))(f). \end{aligned}$$

Hence, we define

$$\mathrm{rec}^{\mathrm{b}} :\equiv \lambda af.\mathrm{ke}(\mathrm{rec}(a)(\lambda k.f(\lambda\alpha.k))).$$

Precook T – relating terms to their b-translations

We define the following parameterized logical relation[3] $\mathrm{R}^\rho_\alpha \subseteq \rho^{\mathrm{b}} \times \rho$ for a given $\alpha : \mathbb{N} \to \mathbb{N}$ by induction on ρ

$$\begin{aligned} f \ \mathrm{R}^{\mathbb{N}}_\alpha \ n \quad &:\equiv \quad f(\alpha) = n \\ g \ \mathrm{R}^{\sigma\to\tau}_\alpha \ h \quad &:\equiv \quad \forall x^{\sigma^{\mathrm{b}}} y^{\sigma} \left(x \ \mathrm{R}^\sigma_\alpha \ y \to g(x) \ \mathrm{R}^\tau_\alpha \ h(y) \right). \end{aligned}$$

Lemma. For any term $t : \rho$ in T, we have

$$t^{\mathrm{b}} \ \mathrm{R}^\rho_\alpha \ t$$

for any $\alpha : \mathbb{N} \to \mathbb{N}$, assuming $x^{\mathrm{b}} \ \mathrm{R}_\alpha \ x$ for all $x \in \mathrm{FV}(t)$.

We define a generic element $\Omega : \mathbb{N}^{\mathrm{b}} \to \mathbb{N}^{\mathrm{b}}$ by $\Omega(f)(\alpha) :\equiv \alpha(f\alpha)$ and have $\Omega \ \mathrm{R}_\alpha \ \alpha$ for any $\alpha : \mathbb{N} \to \mathbb{N}$.

Theorem. For any closed term $f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ in T, we have

$$f^{\mathrm{b}}(\Omega)(\alpha) = f(\alpha)$$

for all $\alpha : \mathbb{N} \to \mathbb{N}$.

[3] M. H. Escardó, *Continuity of Gödel's system T functionals via effectful forcing*, MFPS'2013. Electronic Notes in Theoretical Computer Science 298 (2013), 119–141.

Continuity predicate

Recall that a function $f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ is continuous if

$$\forall \alpha^{\mathbb{N}\to\mathbb{N}}\ \exists m^{\mathbb{N}}\ \forall \beta^{\mathbb{N}\to\mathbb{N}}\ ((\forall i < n.\ \alpha_i = \beta_i) \to f\alpha = f\beta)\,.$$

We define a predicate $\mathrm{C}_\rho \subseteq \rho^{\mathrm{b}}$ by

$$\begin{aligned} \mathrm{C}_{\mathbb{N}}(f) \quad &:\equiv \quad f \text{ is continuous} \\ \mathrm{C}_{\sigma\to\tau}(h) \quad &:\equiv \quad \forall x^{\sigma^{\mathrm{b}}}\,(\mathrm{C}_\sigma(x) \to \mathrm{C}_\tau(hx))\,. \end{aligned}$$

Lemma. For any term $t : \rho$ in T, we have

$$\mathrm{C}_\rho(t^{\mathrm{b}})$$

assuming $\mathrm{C}(x^{\mathrm{b}})$ for all $x \in \mathrm{FV}(t)$.

Proof. By induction on t.
When $t \equiv \mathrm{rec}$, we just need to show that the Kleisli extension preserves C, *i.e.* $\mathrm{C}(\mathrm{ke}(g))$ for any $g : \mathbb{N} \to \rho^{\mathrm{b}}$ with $\forall i.\mathrm{C}(g(i))$, by induction on ρ. □

Main result

Lemma. We have

$$\mathrm{C}_{\mathbb{N}\to\mathbb{N}}(\Omega).$$

Proof. Given a continuous $f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$, the goal is to show that $\Omega(f)$ ($\equiv \lambda\alpha.\alpha(f\alpha)$) is also continuous. Given $\alpha : \mathbb{N} \to \mathbb{N}$, let m be the modulus of continuity of f at α. Then $\max(m, f(\alpha) + 1)$ is a modulus of $\Omega(f)$ at α. □

Theorem. Every T-definable function $f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ is continuous.

Proof. By the previous two lemmas, we know that $f^{\mathrm{b}}(\Omega)$ is continuous. Because f and $f^{\mathrm{b}}(\Omega)$ are pointwise equal and continuity is extensional, f is also continuous. □

T-definable moduli of continuity

Theorem. For any T-definable function $f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$, there is a term $m : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ in T internalizing a modulus of continuity of f.

Idea: combine the construction of continuity moduli with the b-translation.

The type translation $\rho \mapsto \rho^{\mathbf{b}}$ becomes

$$\begin{aligned} \mathbb{N}^{\mathbf{b}} &:\equiv ((\mathbb{N} \to \mathbb{N}) \to \mathbb{N}) \times ((\mathbb{N} \to \mathbb{N}) \to \mathbb{N}) \\ (\sigma \to \tau)^{\mathbf{b}} &:\equiv \sigma^{\mathbf{b}} \to \tau^{\mathbf{b}}. \end{aligned}$$

We write $w \equiv \langle \mathrm{V}_w ; \mathrm{M}_w \rangle$ for $w : \mathbb{N}^{\mathbf{b}}$ and have the term translation $t \mapsto t^{\mathbf{b}}$

$$\begin{aligned} \cdots &:\equiv \cdots \\ 0^{\mathbf{b}} &:\equiv \langle \lambda\alpha.0 ; \lambda\alpha.0 \rangle \\ \mathrm{succ}^{\mathbf{b}} &:\equiv \lambda x.\langle \mathrm{succ} \circ \mathrm{V}_x ; \mathrm{M}_x \rangle \\ \mathrm{rec}^{\mathbf{b}} &:\equiv \mathbf{ke}(\mathrm{rec}(a)(\lambda k.f\langle \lambda\alpha.k ; \lambda\alpha.0 \rangle)) \end{aligned}$$

where the base case of the Kleisli extension becomes

$$\mathbf{ke}^{\mathbb{N}}(g)(f) :\equiv \langle \lambda\alpha.\mathrm{V}_{g(\mathrm{V}_f(\alpha))}(\alpha) ; \lambda\alpha.\max(\mathrm{M}_{g(\mathrm{V}_f(\alpha))}(\alpha), \mathrm{M}_f(\alpha)) \rangle.$$

The generic element becomes

$$\mathbf{\Omega}(f) :\equiv \langle \lambda\alpha.\alpha(\mathrm{V}_f(\alpha)) ; \lambda\alpha.\max(\mathrm{M}_f(\alpha), \mathrm{succ}(\mathrm{V}_f(\alpha))) \rangle.$$

Replacing the continuity predicate

Without changing our b-translation, we can prove various properties/structures of T-definable functions $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ with different predicates:

- $\mathrm{B}_{\mathbb{N}}(f) :\equiv$ there exists a bar-recursion functional controlled by f
  We recover Oliva and Steila's proof[4] of Schwichtenberg's bar recursion closure theorem of type level 0.
- $\mathrm{UC}_{\mathbb{N}}(f) :\equiv f|_{2^{\mathbb{N}}}$ is uniformly continuous
  We prove that the restriction of any T-definable f from $\mathbb{N} \to \mathbb{N}$ to $\mathbb{N} \to \mathbb{2}$ is uniformly continuous.
- $\mathrm{D}_{\mathbb{N}}(f) :\equiv f$ has a dialogue tree representation
  We recover Escardó's result[5] without full model construction.
- ...

The case of function spaces in the above predicates is defined as usual.

[4] P. Oliva and S. Steila, *A direct proof of Schwichtenberg's bar recursion closure theorem*, J. Symbolic Logic 83 (2018), no. 1, 70–83.
[5] M. H. Escardó, *Continuity of Gödel's system T functionals via effectful forcing*, MFPS'2013. Electronic Notes in Theoretical Computer Science 298 (2013), 119–141.

Generalizing the b-translation

The term translation $(t \mapsto t^{\mathrm{b}}) : \rho \to \rho^{\mathrm{b}}$ is independent of the choice of X in the type translation

$$\begin{aligned} \mathbb{N}^{\mathrm{b}} &:\equiv X \to \mathbb{N} \\ (\sigma \to \tau)^{\mathrm{b}} &:\equiv \sigma^{\mathrm{b}} \to \tau^{\mathrm{b}}. \end{aligned}$$

Hence we can generalize our syntactic method to prove properties/structures of T-definable functionals of arbitrary type, provided that we can define a $\Omega : X^{\mathrm{b}}$ with (i) $\Omega\ \mathrm{R}^X_\alpha\ \alpha$ for all $\alpha : X$ and (ii) Ω satisfies the predicate.

- $X \equiv \mathbb{1}$ – the identity translation
- $X \equiv \mathbb{N}$
  Take $\Omega :\equiv \mathrm{id}_{\mathbb{N}}$ and try to prove a variant of Ishihara's $\mathrm{BD}\text{-}\mathbb{N}$ (w.i.p.)
- $X \equiv \mathbb{N} \to \mathbb{N}$
  ...
- $X \equiv \mathbb{N} \to \mathbb{N} \to \mathbb{N}$
  Take $\Omega :\equiv \lambda f g x . x(fx)(gx)$ and recover Oliva and Steila's proof for bar recursion of type level 1.

Summary and ...

- We introduce a syntactic approach to prove continuity and other properties/structures of T-definable functionals.
- We want to extend the method to dependent types, but don't know how.
- Our development is constructive and has been formalized/implemented in Agda to compute moduli of (uniform) continuity.

`http://cj-xu.github.io/agda/TCont/`

Thank you!