Ordinal notations via simultaneous definitions

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Introduction

Our goal is to represent ordinals using data structures that are simple, efficient and convenient (for development in type theory).

Our ordinal notation system is based on Cantor normal form:

$$\alpha = \omega^{a_1} + \cdots + \omega^{a_n} \quad \text{where} \quad a_1 \geq \cdots \geq a_n$$

We define a type of nested decreasing lists simultaneously[1] with their ordering.

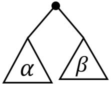
Our ordinal terms are in one-to-one correspondence with the ordinals below ε_0.

Choosing the binary list structure ($[] \mid x :: xs$), a nested list is a binary tree.

[1] Prof. Schwichtenberg in his 2013 course *Selected Topics in Proof Theory* defined ordinal notations with the same idea.

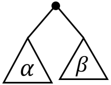
Binary trees as ordinals[2]

1. The leaf represents 0.

2.1 

represents $\omega^{\alpha} \# \beta$ (the Hessenberg sum).

This corresponds to nested finite multisets which can be easily implemented in univalent type theory using higher inductive types (joint work in progress with Fredrik Nordvall Forsberg and Nicolai Kraus).

2.2 

represents $\omega^{\alpha} + \beta$.

Our simultaneous definition generates only the ordered binary trees.

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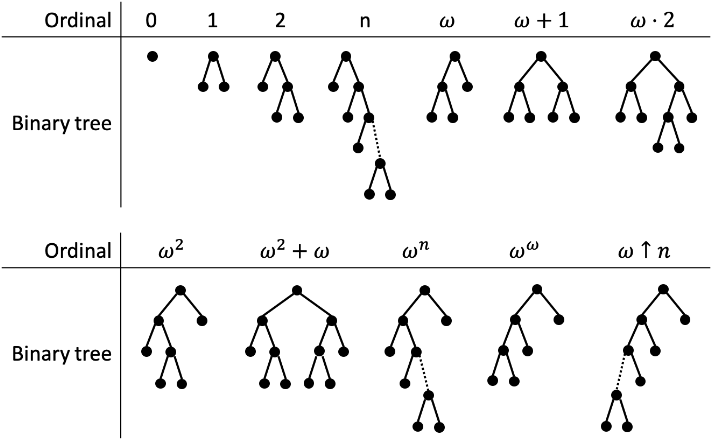
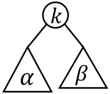
[2] Nachum Dershowitz. Trees, ordinals and termination. In M.C. Gaudel and J.P. Jouannaud, editors, *Theory and Practice of Software Development (TAPSOFT 1993)*, volume 668 of *Lecture Notes in Computer Science*, pages 243–250, 1993.

Examples of binary trees representing ordinals[3]



[3] The diagram is taken from Dershowitz's TAPSOFT'93 paper.

Labeled binary trees as ordinals

1. The leaf represents 0.
2. 

represents $\omega^{\alpha} \cdot k + \beta$.

- The ordered ones correspond to the ordinals in Cantor normal form

$$\alpha = \omega^{a_1} \cdot k_1 + \cdots + \omega^{a_n} \cdot k_n \quad \text{where } a_1 > \cdots > a_n \text{ and } k_i > 0$$

- Such notation system has been implemented in ACL2[4] and Coq[5].

[4] Panagiotis Manolios and Daron Vroon. Ordinal arithmetic: algorithms and mechanization. *Journal of Automated Reasoning*, 34(4):387–423, 2005.

[5] José Grimm. Implementation of three types of ordinals in Coq. Technical Report RR-8407, INRIA, 2013. Available at `https://hal.inria.fr/hal-00911710`.

Idea: ordinals as nested decreasing lists

Each ordinal can be uniquely written as the sum

$$\omega^{a_1} + \cdots + \omega^{a_n}$$

where $a_1 \geq \cdots \geq a_n$ are ordinals, and hence can be equivalently represented by the decreasing list $[a_1, \ldots, a_n]$.

Because the elements of such a decreasing list are also decreasing lists, an ordering on them has to be defined simultaneously.

Moreover, when inserting a new element a in front of an already constructed list as, we need to ensure that a is greater than or equal to the head of as in order to not violate the order invariant, which requires us to also simultaneously define a head function for such lists.

A simultaneous definition of ordinal notations

Definition. We simultaneously define a type $\mathcal{O}$, a binary relation $<$ over $\mathcal{O}$ and a function $\mathrm{fst} : \mathcal{O} \to \mathcal{O}$ as follows

- The type $\mathcal{O}$ is defined with two cases
  ($\mathcal{O}1$) $0 : \mathcal{O}$,
  ($\mathcal{O}2$) if $a, b : \mathcal{O}$ with $a \geq \mathrm{fst}(b)$, then $(a, b) : \mathcal{O}$.
  If a, b denote some ordinals, then (a, b) denotes the ordinal $\omega^a + b$.
  Hence we usually write the latter instead of pairs.

- The relation $<$ is inductively given by the following clauses
  (< 1) If $a \neq 0$ then $0 < a$.
  (< 2) If $a < b$ then $\omega^a + c < \omega^b + d$.
  (< 3) If $b < c$ then $\omega^a + b < \omega^a + c$.

- The function $\mathrm{fst} : \mathcal{O} \to \mathcal{O}$ is defined by cases on its argument

$$\begin{aligned} \mathrm{fst}(0) &:\equiv 0 \\ \mathrm{fst}(\omega^a + b) &:\equiv a. \end{aligned}$$

Properties and examples

Proposition. The relation $<$ is a strict total order on $\mathcal{O}$.

Proof. Irreflexivity and transitivity are proved by structural induction on the relation. Trichotomy is proved by induction on the arguments.

Lemma. $a \geq 0$ for all $a : \mathcal{O}$.

Proof. By induction on a.

Examples. $\omega^a :\equiv \omega^a + 0$ $\quad 1 :\equiv \omega^0 \equiv \omega^0 + 0$ $\quad \omega :\equiv \omega^1 \equiv \omega^{\omega^0 + 0} + 0$

Because each ordinal term $a : \mathcal{O}$ is a list, we can compute its length $|a|$ and access its elements $a_i : \mathcal{O}$ as for lists. We define

- $\mathrm{isFin}(a) :\equiv a_0 = 0$
- $\mathrm{isSuc}(a) :\equiv \exists_{i<|a|}\, a_i = 0$
- $\mathrm{isLim}(a) :\equiv a \neq 0 \wedge a_{|a|-1} \neq 0$

Embedding into set-theoretic ordinals

Definition. For each ordinal term $a : \mathcal{O}$, we associate a set-theoretic ordinal $[\![a]\!]$

$$\begin{aligned} [\![0]\!] &:\equiv 0 \\ [\![\omega^a + b]\!] &:\equiv \omega^{[\![a]\!]} + [\![b]\!]. \end{aligned}$$

Lemma. For any $a, b : \mathcal{O}$, we have $a < b$ iff $[\![a]\!] < [\![b]\!]$.

Proposition. Let $a : \mathcal{O}$. We have

1. $[\![a]\!]$ is in Cantor normal form;
2. $[\![a]\!]$ is finite iff $\mathrm{isFin}(a)$ holds;
3. $[\![a]\!]$ is a successor iff $\mathrm{isSuc}(a)$ holds;
4. $[\![a]\!]$ is a limit iff $\mathrm{isLim}(a)$ holds.

Theorem

1. For any $a : \mathcal{O}$, we have $[\![a]\!] < \varepsilon_0$.
2. For every ordinal $\alpha < \varepsilon_0$ there exists an ordinal term $a : \mathcal{O}$ with $\alpha = [\![a]\!]$.

Addition

Because of the mutual definitions, we have to simultaneously
(i) define addition of $\mathcal{O}$ and (2) prove a property of the addition.

Definition. Addition of $\mathcal{O}$ is defined as follows

$$
\begin{aligned}
0 + b &:\equiv b \\
a + 0 &:\equiv a \\
(\omega^a + c) + (\omega^b + d) &:\equiv \begin{cases} \omega^b + d & \text{if } a < b \\ \omega^a + (c + (\omega^b + d)) & \text{otherwise} \end{cases}
\end{aligned}
$$

The last case is well-defined because $a \geq \mathrm{fst}(c + (\omega^b + d))$ by

Lemma. For any $a, b, c : \mathcal{O}$, if $a \geq \mathrm{fst}(b)$ and $a \geq \mathrm{fst}(c)$ then $a \geq \mathrm{fst}(b + c)$.

Proof. By induction on b and c.

Theorem. $[\![a + b]\!] = [\![a]\!] + [\![b]\!]$.

Subtraction

Definition. Subtraction of $\mathcal{O}$ is defined as follows

$$
\begin{aligned}
0 - b &:\equiv 0 \\
a - 0 &:\equiv a \\
(\omega^a + c) - (\omega^b + d) &:\equiv \begin{cases} 0 & \text{if } a < b \\ c - d & \text{if } a = b \\ \omega^a + c & \text{otherwise.} \end{cases}
\end{aligned}
$$

Lemma. For any $a, b : \mathcal{O}$, if $a \leq b$ then $a - b = 0$; otherwise, $b + (a - b) = a$.

Theorem. $[\![a - b]\!] = [\![a]\!] - [\![b]\!]$.

Multiplication

Definition. Multiplication of $\mathcal{O}$ is defined as follows

$$\begin{aligned} 0 \cdot b &:\equiv 0 \\ a \cdot 0 &:\equiv 0 \\ a \cdot (\omega^0 + d) &:\equiv a + a \cdot d \\ (\omega^a + c) \cdot (\omega^b + d) &:\equiv \omega^{a+b} + (\omega^a + c) \cdot d \qquad (b \neq 0). \end{aligned}$$

The last case is defined using the following facts:

- Ordinal multiplication is distributive on the left;
- $(\omega^\alpha + \omega^\gamma) \cdot \omega^\beta = \omega^{\alpha+\beta}$ for any ordinals $\alpha \geq \gamma$ and $\beta \neq 0$.

Theorem. $[\![a \cdot b]\!] = [\![a]\!] \cdot [\![b]\!]$.

Exponentiation

Definition. Exponentiation of $\mathcal{O}$ is defined as follows

$$
\begin{aligned}
a^{0} &:\equiv 1 & \\
0^{b} &:\equiv 0 & (b > 0) \\
1^{b} &:\equiv 1 & \\
a^{\omega^{0} + d} &:\equiv a \cdot a^{d} & (a > 1) \\
(\omega^{0} + c)^{\omega^{1} + d} &:\equiv \omega \cdot (\omega^{0} + c)^{d} & (c > 0) \\
(\omega^{0} + c)^{\omega^{b} + d} &:\equiv \omega^{b-1} \cdot (\omega^{0} + c)^{d} & (b > 1 \ \& \ c > 0) \\
(\omega^{a} + c)^{\omega^{b} + d} &:\equiv \omega^{a \cdot \omega^{b}} \cdot (\omega^{a} + c)^{d} & (a > 0 \ \& \ b > 0).
\end{aligned}
$$

The last three cases are defined using the following facts:

- $k^{\omega} = \omega$ for any finite ordinal k;
- $k^{\omega^{\alpha}} = \omega^{\alpha - 1}$ for finite k and ordinal $\alpha > 1$;
- $(\omega^{\alpha} + \omega^{\gamma})^{\omega^{\beta}} = \omega^{\alpha \cdot \omega^{\beta}}$ for any ordinals $\alpha, \beta > 0$ and $\gamma \leq \alpha$.

Theorem. $[\![a^{b}]\!] = [\![a]\!]^{[\![b]\!]}$.

Agda implementation

Available at http://cj-xu.github.io/agda/ordinals/index.html

Including:

- an inductive-inductive-recursive definition of $\mathcal{O}$, using Agda's irrelevance
- a show function to print elements of $\mathcal{O}$ in a readable way
- a proof that $<$ is a strict total order on $\mathcal{O}$
- arithmetic of $\mathcal{O}$
- some examples for computation
- an equivalent inductive-inductive-inductive definition of $\mathcal{O}$

Summary and ...

We have presented a simultaneous definition of ordinal notations that are in one-to-one correspondence with the ordinals below ε_0, which has been implemented in Agda as an inductive-inductive-recursive definition.

Future tasks:

- formalize the correctness proofs in Agda,
- prove the well-foundedness of $\mathcal{O}$,
- implement proof(s) of proof-theoretic strength, theory expressivity or program termination in Agda using $\mathcal{O}$,
- generalize the method to represent ordinals beyond ε_0.

Thank you!