A constructive model of uniform continuity

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Uniform continuity axiom

$$\forall f\colon 2^{\mathbb{N}} \to \mathbb{N} \quad \exists n \in \mathbb{N} \quad \forall \alpha, \beta \in 2^{\mathbb{N}} \, (\alpha =_n \beta \implies f(\alpha) = f(\beta)).$$

1. $\boxed{\text{Provable}}$ in Brouwerian intuitionistic mathematics (INT).

2. $\boxed{\text{Independent}}$ of

   - Bishop's mathematics (BISH),
   - higher-type Heyting arithmetic (HA^ω),
   - Martin-Löf's type theory (MLTT), ...

   Becomes provably false if excluded middle is postulated.
   Becomes provably true if Brouwerian axioms are postulated.

3. $\boxed{\text{Provably false}}$ in Markov's constructive recursive mathematics (RUSS).

Goal

Constructively build a model of some forms of constructive mathematics, including BISH, HA^ω, MLTT (and of course excluding RUSS):

1. in which the uniform continuity axiom holds,

2. without assuming continuity axioms or any constructively contentious axiom in the meta-language.

Consequence

In varieties of constructive mathematics such as BISH, HA^ω, MLTT:

1. Any definable function is uniformly continuous.
2. There is an algorithm that from a definition of a function builds a proof of its uniform continuity.
3. The algorithm doesn't need to be written down explicitly.
4. It is implicit in the construction of the model and the definition of the semantics of the formal system.

We have done this for HA^ω, and we intend to do this for MLTT.

Models of uniform continuity

Mike Fourman (1982) constructed sheaf models of uniform continuity.

$\boxed{\text{Kripke–Joyal semantics}}$ for the quantifiers $\forall, \exists$.

"Local truth."

We instead want $\boxed{\text{Brouwer–Heyting–Kolmogorov semantics}}$ for the quantifiers:

$$\prod_{f\,:\,2^{\mathbb{N}} \to \mathbb{N}} \; \sum_{n\,:\,\mathbb{N}} \; \prod_{\alpha,\beta\,:\,2^{\mathbb{N}}} (\alpha =_n \beta \implies f(\alpha) = f(\beta)).$$

Realizability interpretation.

Idea

1. Consider sheaves and natural transformations like Mike Fourman.
2. Develop this in a meta-language with a BHK interpretation of logic.

We will get realizability out of the meta-language, rather than out of the model.

We will be able to compute our claims.

But we will not need to consider notions of computability to build the model.

The capability of performing computations is built-in in our meta-language.

Natural choice of a meta-language

Martin-Löf type theory with a universe.

1. Sufficiently powerful.
2. BHK semantics of logic.
3. Implemented as a subset of various systems such as Coq, Lego, Agda.

Precursors of our work include

0. Spanier's quasi-topological spaces (1961).
1. Johnstone's paper On a topological topos (1979).
2. Fourman's papers Continuous truth and Notions of choice sequence (1982).
3. van der Hoeven and Moerdijk's paper Sheaf models for choice sequences (1984).
4. Bauer and Simpson's unpublished work Continuity begets continuity (2006).

Technical motivation: Spanier's quasi-topological spaces

Def. A quasi-topology on a set X assigns to each compact Hausdorff space K, a set $P(K, X)$ of "probes" $p\colon K \to X$, such that:

1. All constant maps are in $P(K, X)$.
2. If $t\colon K' \to K$ is continuous and $p \in P(K, X)$, then $p \circ t \in P(K', X)$. (Presheaf condition.)
3. If $(t_i\colon K_i \to K)_{i \in I}$ is finite, jointly surjective family and $p\colon K \to X$ is a function with $p \circ t_i \in P(K_i, X)$ for every $i \in I$, then $p \in P(K, X)$. (Sheaf condition.)

Def. A function $f\colon X \to Y$ of quasi-topological spaces is continuous if $p \in P(K, X)$ implies $f \circ p \in P(K, Y)$.
(Naturality condition.)

Facts

1. Quasi-topological spaces form a cartesian closed category (Spanier 1961).

2. Quasi-topological spaces form a quasi-topos (Dubuc 1970's).

3. The quasi-topological spaces arise as the "concrete sheaves" of a topos.

Variations

1. Rather than all compact Hausdorff spaces, consider only one.

   $\boxed{\mathbb{N}_\infty\text{, the one-point compactification of discrete }\mathbb{N}.}$

   The quasi-spaces are simply the Kuratowski limit spaces.

   Johnstone's topological topos enlarges this.

   Topos considered by Bauer and Simpson.

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2. Again consider only one space, but forget about compactness.

   $\boxed{\mathbb{N}^{\mathbb{N}}\text{, the Baire space.}}$

   Topos considered by Fourman, and by van der Hoeven and Moerdijk.

Our variation

$\boxed{\text{Consider only one compact Hausdorff space, the Cantor space } 2^{\mathbb{N}}.}$

Classically, we should get the same thing as the topological topos.

We now describe the sheaves in a way suitable for treatment in MLTT.

Underlying category of the site

$$\boxed{\text{The monoid } C \text{ of uniformly continuous maps } t\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}.}$$

Rather than the category of continuous maps of compact Hausdorff spaces.

It is vital here that we say "uniformly continuous".

Rather than just "continuous" and rely on the "fact" that continuous functions are uniformly continuous.

This is because our meta-language cannot prove this "fact".

Presheaves

A presheaf can be described as a set P equipped with an action

$$\begin{array}{rcl} P \times C & \to & P \\ (p, t) & \mapsto & p \cdot t \end{array}$$

satisfying

$$\begin{array}{rcl} p \cdot \mathrm{id} & = & p, \\ p \cdot (t \circ u) & = & (p \cdot t) \cdot u. \end{array}$$

Natural transformation

A natural transformation of presheaves $(P, \cdot)$ and $(Q, \cdot)$ is a function $f\colon P \to Q$ that preserves the action:

$$f(x \cdot t) = (fx) \cdot t.$$

The coverage

1. Let 2^n denote the set of binary strings of length n.

2. For $s \in 2^n$, let $\mathrm{cons}_s \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ denote the concatenation map

$$\mathrm{cons}_s(\alpha) = s\alpha.$$

For each natural number n we have the covering family $(\mathrm{cons}_s)_{s \in 2^n}$.

1. Jointly surjective.

2. Disjoint images (simplifies the sheaf condition).

Sheaves

A presheaf $(P, \cdot)$ is a sheaf if and only if

For any n and any family $(p_s \in P)_{s \in 2^n}$ (the compatibility condition always holds), there is a unique $p \in P$ with

$$p \cdot \mathrm{cons}_s = p_s.$$

It suffices to check the case $n = 1$

A presheaf $(P, \cdot)$ is a sheaf if and only if

For any two $p_0, p_1 \in P$ there is a unique $p \in P$ with

$$p \cdot \mathrm{cons}_0 = p_0, \qquad p \cdot \mathrm{cons}_1 = p_1$$

Theorem of MLTT with a universe

Our category of sheaves is cartesian closed.

And even locally cartesian closed.

The usual constructions of products and exponentials work.

But (provably) it can't have a subobject classifier, because MLTT itself doesn't.

So MLTT cannot prove it is a topos.

Concrete sheaf

A sheaf $(P, \cdot)$ where the action $(\cdot)$ is function composition.

Then P must be a set of maps $2^{\mathbb{N}} \to X$ for some X.

Concrete sheaves can be described as follows

A set X equipped with a set P of "probes" $2^{\mathbb{N}} \to X$ such that

1. All constant maps are in P.
2. If $t\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is continuous and $p \in P$, then $p \circ t \in P$.
   (Presheaf condition.)
3. For any two $p_0, p_1 \in P$, the unique function $p\colon 2^{\mathbb{N}} \to X$ defined by $p(i * \alpha) = p_i(\alpha)$ is in P.
   (Sheaf condition.)

A concrete natural transformation of concrete sheaves amounts to a function $f\colon X \to Y$ such that $f \circ p \in P_Y$ whenever $p \in P_X$.

Concrete sheaves are sheaves

A concrete sheaf is a set X equipped with a set P of probes $2^{\mathbb{N}} \to X$ s.t. ...

A concrete natural transformation of concrete sheaves then amounts to a function $f\colon X \to Y$ such that $f \circ p \in P_Y$ whenever $p \in P_X$.

From the concrete sheaf (X, P) we get the sheaf $(P, \circ)$.

From the concrete sheaf (Y, Q) we get the sheaf $(Q, \circ)$.

From a concrete natural transformation $f\colon X \to Y$ as above we get the natural transformation $\phi\colon P \to Q$ defined by

$$\phi(p) = f \circ p$$

and all natural transformations of concrete sheaves arise in this way.

As is well known

Every sheaf topos has a natural numbers object.

It is the sheafification of natural numbers object in the presheaf topos.

Which in turn is the constant presheaf whose values are the natural numbers set.

Trouble. The sheafification process seems to be impredicative.

But we can still show in our predicative meta-language that the category of sheaves does have a natural numbers object.

Theorem of MLTT with a universe

Our category of sheaves has a natural numbers object, which is a concrete sheaf.

It is the type $\mathbb{N}$ of natural numbers equipped with the set of uniformly continuous functions $2^{\mathbb{N}} \to \mathbb{N}$, with zero and successor inherited from the type $\mathbb{N}$.

The Yoneda Embedding

Regard the underlying monoid C of our site as a one-object full subcategory of a category of spaces, with object denoted by $2^{\mathbb{N}}$.

We have that $y(2^{\mathbb{N}}) = (C, \circ)$.

We also have $y(2^{\mathbb{N}}) \cong 2^{\mathbb{N}}$ where in the righthand side:

- 2 is $1 + 1$ calculated in the category of sheaves with 1 the terminal sheaf,
- $\mathbb{N}$ is the natural numbers object constructed earlier,
- the exponential is calculated in the category of sheaves.

In particular, this shows that our coverage is sub-canonical.

The Yoneda Lemma

1. If $A = (P, \cdot)$ is a sheaf, then

$$\big(\text{natural transformations } y(2^{\mathbb{N}}) \to A\big) \cong P.$$

2. Using $y(2^{\mathbb{N}}) \cong 2^{\mathbb{N}}$ and $\mathbb{N} = (\text{uniformly continuous functions } 2^{\mathbb{N}} \to \mathbb{N}, \circ)$, we get

$$\big(\text{natural transformations } 2^{\mathbb{N}} \to \mathbb{N}\big) \cong \text{uniformly continuous functions } 2^{\mathbb{N}} \to \mathbb{N}.$$

3. From this we get a natural transformation

$$\mathrm{fan}\colon (2^{\mathbb{N}} \to \mathbb{N}) \to \mathbb{N}$$

in the model that calculates moduli of uniform continuity.

Future work

1. We plan to account for MLTT itself in the future, although this will require a slightly more powerful meta-language, of course.

2. We plan to look at other things such as Bar induction.

   As in Fourman's and in van der Hoeven and Moerdijk's work.

3. We are also implementing the above development in the programming language/proof system Agda.

   This will allow us to run the proofs and extract proofs of uniform continuity, and illustrate the practical advantages of using a constructive meta-language.