Towards an implementation of
a functional interpretation for nonstandard arithmetic

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Abstract

We aim to develop the nonstandard Dialectica interpretation [4] and its soundness proof in Agda [3], so that we can extract Agda programs from nonstandard proofs. This note recalls the definition of a constructive nonstandard arithmetic, namely system H, and formulates the nonstandard Dialectica interpretation in a way that is suitable for a formal, type-theoretic development.

1 Nonstandard arithmetic H

We recall the definition of the constructive nonstandard system H introduced in [4] which is a conservative extension of $\mathsf{E\text{-}HA}^\omega$, the Heyting arithmetic with finite types.

Term language. The term language T^* of System H is an extension of Gödel's T with finite sequences. Specifically, it has a base type $\mathbb{N}$ of natural numbers, function types $\sigma \to \tau$, and sequence types σ^*:

(i) For $\mathbb{N}$, we have two constructors $0 : \mathbb{N}$ and $\mathsf{s} : \mathbb{N} \to \mathbb{N}$, and a recursor $\mathsf{Rec}_\sigma : \sigma \to (\mathbb{N} \to \sigma \to \sigma) \to \mathbb{N} \to \sigma$ for each σ, satisfying

$$\begin{aligned} \mathsf{Rec}_\sigma(a, f, 0) &=_\sigma a \\ \mathsf{Rec}_\sigma(a, f, \mathsf{s}n) &=_\sigma f(n, \mathsf{Rec}_\sigma(a, f, n)) \end{aligned}$$

for any $a : \sigma$, $f : \mathbb{N} \to \sigma \to \sigma$ and $n : \mathbb{N}$.

(ii) For function types, we have lambda abstraction $\lambda x^\sigma.t : \sigma \to \tau$ for $t : \tau$, and application $fa : \tau$ for $f : \sigma \to \tau$ and $a : \sigma$, satisfying the β- and η-rules.

(iii) For sequence types, we have two constructors $[] : \sigma^*$ (the empty sequence) and $\_ :: \_ : \sigma \to \sigma^* \to \sigma^*$ (the prepending operation) and a list recursor $\mathsf{Rec}^*_{\sigma,\tau} : \sigma \to (\tau \to \sigma \to \sigma) \to \tau^* \to \sigma$ for each σ and τ, satisfying

$$\begin{aligned} \mathsf{Rec}^*_{\sigma,\tau}(a, f, []) &=_\sigma a \\ \mathsf{Rec}^*_{\sigma,\tau}(a, f, x :: xs) &=_\sigma f(x, \mathsf{Rec}_\sigma(a, f, xs)) \end{aligned}$$

for any $a : \sigma$, $f : \tau \to \sigma \to \sigma$, $x : \tau$ and $xs : \tau^*$.

Using the list recursor, we can define a length function $|\_| : \sigma^* \to \mathbb{N}$ such that, for all $x : \sigma$ and $xs : \sigma^*$,

$$\begin{aligned} |[]| &=_\mathbb{N} 0 \\ |x :: xs| &=_\mathbb{N} \mathsf{s}(|xs|) \end{aligned}$$

and a projection function $(xs, i) \mapsto xs_i$ of type $\sigma^* \to \mathbb{N} \to \sigma$ such that, for all $x : \sigma$, $xs : \sigma^*$ and $i : \mathbb{N}$,

$$\begin{aligned} []_i &=_\sigma \mathbf{c}^\sigma \\ (x :: xs)_0 &=_\sigma x \\ (x :: xs)_{\mathsf{s}i} &=_\sigma xs_i \end{aligned}$$

where $\mathbf{c}^\sigma$ is a canonically chosen element of σ (notice that all types in T^* are non-empty). Both functions will be used for formulating the axiom of extensionality for sequence types.

Logic. We firstly have a theory $\mathsf{E\text{-}HA}^{\omega*}$ by extending $\mathsf{E\text{-}HA}^\omega$ with the axioms and rules for finite sequences. Then we have $\mathsf{E\text{-}HA}^{\omega*}_{\mathsf{st}}$ by extending $\mathsf{E\text{-}HA}^{\omega*}$ with unary predicates st^σ for each type σ. And system H is an extension of $\mathsf{E\text{-}HA}^{\omega*}_{\mathsf{st}}$ with a few nonstandard principles.

Specifically, *formulas* of $\mathsf{E\text{-}HA}^{\omega*}_{\mathsf{st}}$ (and thus of H) are built up as follows:

(1) Equations $a =_\sigma b$, where terms $a, b : \sigma$ have the same type, are atomic formulas.

(2) For each term $t : \sigma$ we have an atomic formula $\mathsf{st}^\sigma(t)$.

(3) If A and B are formulas, then so are $A \wedge B$, $A \vee B$ and $A \Rightarrow B$.

(4) If A is a formula and x^σ is a variable, then $\forall x^\sigma A$, $\exists x^\sigma A$, $\forall^{\mathrm{st}} x^\sigma A$ and $\exists^{\mathrm{st}} x^\sigma A$ are formulas.

And we adopt the following abbreviations:

(a) $n < m \ := \ \exists k \ \mathsf{s}n + k = m$

(b) $\forall i{<}n \ A \ :\equiv \ \forall i \ (i{<}n \Rightarrow A)$
$\exists i{<}n \ A \ :\equiv \ \exists i \ (i{<}n \wedge A)$

(c) $x \in_\sigma xs \ := \ \exists i{<}|xs| \ x =_\sigma xs_i$

(d) $\forall x \in_\sigma xs \ A(x) \ := \ \forall i{<}|xs| \ A(xs_i)$
$\exists x \in_\sigma xs \ A(x) \ := \ \exists i{<}|xs| \ A(xs_i)$

We may omit the superscribed and subscribed types if no confusion is caused.

Since we have a primitive notion of equality at each type, we assume the following axioms of extensionality:

- $f =_{\sigma\to\tau} g \ \Leftrightarrow \ \forall x^\sigma \ fx =_\tau gx$
- $xs =_{\sigma^*} ys \ \Leftrightarrow \ (|xs| =_{\mathbb{N}} |ys|) \wedge (\forall i{<}|xs| \ xs_i =_\sigma ys_i)$

A formula is called *internal* if it does not contain st; otherwise, it is called *external*. We use Φ, Ψ for arbitrary formulas and φ, ψ for internal formulas in this note.

$\mathsf{E\text{-}HA}^{\omega *}$ extends $\mathsf{E\text{-}HA}^{\omega}$ with the following *sequence axiom*

- $\mathsf{SA}: \ \forall xs^{\sigma^*}(xs =_{\sigma^*} []_\sigma \vee \exists x^\sigma, ys^{\sigma^*} \ xs =_{\sigma^*} x{::}ys)$

as well as the two equations for the list recursor Rec^* introduced above.

$\mathsf{E\text{-}HA}^{\omega *}_{\mathsf{st}}$ has all the axioms and rules from $\mathsf{E\text{-}HA}^{\omega *}$. The ones of intuitionistic logic apply to *all* formulas. Moreover, it has the following axioms for st:

- $\mathsf{st}(x) \wedge x =_\sigma y \Rightarrow \mathsf{st}(y)$ for any terms $x, y : \sigma$.
- $\mathsf{st}(t)$ for all closed term t.
- $\mathsf{st}(f) \wedge \mathsf{st}(x) \Rightarrow \mathsf{st}(fx)$.
- $\forall^{\mathsf{st}} x^\sigma A(x) \Leftrightarrow \forall x^\sigma (\mathsf{st}(x) \Rightarrow A(x))$.
- $\exists^{\mathsf{st}} x^\sigma A(x) \Leftrightarrow \exists x^\sigma (\mathsf{st}(x) \wedge A(x))$.

There are two induction axioms in $\mathsf{E\text{-}HA}^{\omega *}_{\mathsf{st}}$. The *internal* one IA applies to *internal* formulas only, while the *external* one $\mathsf{IA}^{\mathsf{st}}$ applies to *arbitrary* formulas:

- $\mathsf{IA}: \quad \varphi(0) \wedge \forall n \ (\varphi(n) \Rightarrow \varphi(\mathsf{s}n)) \ \Rightarrow \ \forall n \ \varphi(n)$
- $\mathsf{IA}^{\mathsf{st}}: \ \Phi(0) \wedge \forall^{\mathsf{st}} n \ (\Phi(n) \Rightarrow \Phi(\mathsf{s}n)) \ \Rightarrow \ \forall^{\mathsf{st}} n \ \Phi(n)$

where $\varphi(n)$ is *internal* and $\Phi(n)$ can be any formula.

System H extends $\mathsf{E\text{-}HA}^{\omega *}_{\mathsf{st}}$ with the following principles:

- A higher-type version of Nelson's *idealisation* principle (I) [1]
  $\mathsf{I}: \ \forall^{\mathsf{st}} xs^{\sigma^*} \exists y^\tau \forall x {\in} xs \ \varphi(x, y) \ \Rightarrow \ \exists y^\tau \forall^{\mathsf{st}} x^\sigma \varphi(x, y)$
  where $\varphi(x, y)$ is *internal*.
- The *nonclassical realisation* principle (NCR)
  $\mathsf{NCR}: \ \forall y^\tau \exists^{\mathsf{st}} x^\sigma \Phi(x, y) \ \Rightarrow \ \exists^{\mathsf{st}} xs^{\sigma^*} \forall y^\tau \exists x \in_\sigma xs \, \Phi(x, y)$
  where $\Phi(x, y)$ can be *any* formula.
- The *herbrandised axiom of choice* (HAC)
  $\mathsf{HAC}: \ \forall^{\mathsf{st}} x^\sigma \exists^{\mathsf{st}} y^\tau \Phi(x, y) \ \Rightarrow \ \exists^{\mathsf{st}} F^{\sigma\to\tau^*} \forall^{\mathsf{st}} x^\sigma \exists y {\in} F(x) \, \Phi(x, y)$
  where $\Phi(x, y)$ can be *any* formula.
- A *herbrandised* form of a *generalised Markov's principle* ($\mathsf{HGMP}^{\mathsf{st}}$)
  $\mathsf{HGMP}^{\mathsf{st}}: \ (\forall^{\mathsf{st}} x^\sigma \varphi(x) \Rightarrow \psi) \ \Rightarrow \ \exists^{\mathsf{st}} xs^{\sigma^*} (\forall x {\in} xs \, \varphi(x) \Rightarrow \psi)$
  where both $\varphi(x)$ and ψ are *internal*.
- A *herbrandised independence of premise* principle ($\mathsf{HIP}_{\forall^{\mathsf{st}}}$) for formulas of the form $\forall^{\mathsf{st}} x.\varphi(x)$
  $\mathsf{HIP}_{\forall^{\mathsf{st}}}: \ (\forall^{\mathsf{st}} x^\sigma \varphi(x) \Rightarrow \exists^{\mathsf{st}} y^\tau \Psi(y)) \ \Rightarrow \ \exists^{\mathsf{st}} ys^{\tau^*} (\forall^{\mathsf{st}} x^\sigma \varphi(x) \Rightarrow \exists y {\in} ys \, \Psi(y))$
  where $\varphi(x)$ is *internal* and $\Psi(y)$ can be *any* formula.

2 A functional interpretation for H

The idea of the nonstandard Dialectica interpretation for system H, introduced in [4], is that each formula Φ is interpreted by $\exists^{\mathrm{st}} r \forall^{\mathrm{st}} u |\Phi|^r_u$ where $|\Phi|^r_u$ is an internal formula and r is a finite sequence of *potential* realisers, of which at least one is the actual realiser. In this note, we formulate it in a way that is suitable for a formal, type-theoretic development. For this, we figure out the types of realisers and counterexamples which are omitted in [4]. And we add binary products to $\mathsf{HA}^{\omega *}_{\mathsf{st}}$ rather than tuples as in [4] to simplify the typing.

Here are some notations of sequences we use in this section for convenience:

- Given $a : \sigma$, we write $[a^n] : \sigma^*$ for the sequence containing n copies of a, and particularly $[a]$ for the singleton.
- Given $xs, ys : \sigma^*$, we write $xs \mathbin{++} ys$ for the concatenation of xs and ys defined using the list recursor.
- Given $w : (\sigma \times \tau)^*$, we have sequences $w^1 : \sigma^*$ and $w^2 : \tau^*$ (by mapping the projection functions to w).
- Given $F : (\sigma \to \tau^*)^*$ and $x : \sigma$, we write $F[x] : \tau^*$ for the sequence $F_0(x) \mathbin{++} F_1(x) \mathbin{++} \dots \mathbin{++} F_{|F|-1}(x)$.
- Given $F : (\sigma \to \tau \to \rho^*)^*$, $x : \sigma$ and $y : \tau$, we write $F[x, y] : \rho^*$ for the sequence $F_0(x, y) \mathbin{++} \dots \mathbin{++} F_{|F|-1}(x, y)$.

Since every type is inhabited, we choose a canonical element $\mathbf{c}^\sigma$ for each type by induction on σ; in particular, we choose $\mathbf{c}^{\sigma^*} :\equiv [\mathbf{c}^\sigma]$ (*i.e.* the singleton sequence rather than the empty one).

Definition 1. We simultaneously associate to each formula A types $\mathsf{d}^+(A)$ of (*actual*) *realisers* and $\mathsf{d}^-(A)$ of *counterexamples*:

$$
\begin{array}{ll}
\mathsf{d}^+(a =_\sigma b) :\equiv \mathbb{1} & \mathsf{d}^-(a =_\sigma b) :\equiv \mathbb{1} \\
\mathsf{d}^+(\mathsf{st}^\sigma(t)) :\equiv \sigma & \mathsf{d}^-(\mathsf{st}(t)) :\equiv \mathbb{1} \\
\mathsf{d}^+(A \wedge B) :\equiv \mathsf{d}^+A \times \mathsf{d}^+B & \mathsf{d}^-(A \wedge B) :\equiv \mathsf{d}^-A \times \mathsf{d}^-B \\
\mathsf{d}^+(A \vee B) :\equiv \mathsf{d}^+A \times \mathsf{d}^+B & \mathsf{d}^-(A \vee B) :\equiv \mathsf{d}^-A \times \mathsf{d}^-B \\
\mathsf{d}^+(A \Rightarrow B) :\equiv ((\mathsf{d}^+A)^* \to (\mathsf{d}^+B)^*) \times ((\mathsf{d}^+A)^* \to \mathsf{d}^-B \to (\mathsf{d}^-A)^*) & \mathsf{d}^-(A \Rightarrow B) :\equiv (\mathsf{d}^+A)^* \times \mathsf{d}^-B \\
\mathsf{d}^+(\forall x^\sigma A) :\equiv \mathsf{d}^+A & \mathsf{d}^-(\forall x^\sigma A) :\equiv \mathsf{d}^-A \\
\mathsf{d}^+(\exists x^\sigma A) :\equiv \mathsf{d}^+A & \mathsf{d}^-(\exists x^\sigma A) :\equiv (\mathsf{d}^-A)^* \\
\mathsf{d}^+(\forall^{\mathrm{st}} x^\sigma A) :\equiv \sigma \to (\mathsf{d}^+A)^* & \mathsf{d}^-(\forall^{\mathrm{st}} x^\sigma A) :\equiv \sigma \times \mathsf{d}^-A \\
\mathsf{d}^+(\exists^{\mathrm{st}} x^\sigma A) :\equiv \sigma \times \mathsf{d}^+A & \mathsf{d}^-(\exists^{\mathrm{st}} x^\sigma A) :\equiv (\mathsf{d}^-A)^*
\end{array}
$$

Then, for every formula A and terms $r : (\mathsf{d}^+A)^*$ and $u : \mathsf{d}^-A$, we define an *internal* formula $|A|^r_u$ by induction on A:

(i) $|a =_\sigma b|^r_u \;:\equiv\; a =_\sigma b$

(ii) $|\mathsf{st}^\sigma(t)|^r_u \;:\equiv\; t \in_\sigma r$

(iii) $|A \wedge B|^r_u \;:\equiv\; |A|^{r^1}_{u_1} \wedge |B|^{r^2}_{u_2}$

(iv) $|A \vee B|^r_u \;:\equiv\; |A|^{r^1}_{u_1} \vee |B|^{r^2}_{u_2}$

(v) $|A \Rightarrow B|^W_{r,v} \;:\equiv\; \forall u \in W^2[r, v]\, |A|^r_u \Rightarrow |B|^{W^1[r]}_v$

(vi) $|\forall z^\sigma \Phi(z)|^r_u \;:\equiv\; \forall z^\sigma |\Phi(z)|^r_u$

(vii) $|\exists z^\sigma \Phi(z)|^r_u \;:\equiv\; \exists z^\sigma \forall v \in u\, |\Phi(z)|^r_v$

(viii) $|\forall^{\mathrm{st}} z^\sigma \Phi(z)|^R_{a,u} \;:\equiv\; |\Phi(a)|^{R[a]}_u$

(ix) $|\exists^{\mathrm{st}} z^\sigma \Phi(z)|^r_u \;:\equiv\; \exists z \in r^1 \forall v \in u\, |\Phi(z)|^{r^2}_v$

The *nonstandard Dialectica interpretation* $\mathsf{D}_{\mathsf{st}}(\Phi)$ of a formula Φ is defined by

$$\mathsf{D}_{\mathsf{st}}(\Phi) \;:\equiv\; \exists^{\mathrm{st}} r^{(\mathsf{d}^+\Phi)^*} \forall^{\mathrm{st}} u^{\mathsf{d}^-\Phi} |\Phi|^r_u.$$

Remark. (1) In the definition of $\mathsf{d}^{+/-}$, if we ignore the cases of $\mathsf{st}, \forall, \exists$, treat $\forall^{\mathrm{st}}, \exists^{\mathrm{st}}$ as $\forall, \exists$, and remove $*$, then what we get is exactly the types of realisers and counterexamples in the Dialectica interpretation (see *e.g.* [2, §7.4]). (2) A realiser of a formula Φ is a sequence (*i.e.* a term of type $(\mathsf{d}^+\Phi)^*$) containing at least one *actual* realiser (which is a term of type $\mathsf{d}^+\Phi$). (3) A counterexample of Φ is simply a term of type $\mathsf{d}^-\Phi$, as counterexamples are universally quantified in the D_{st}-interpretation.

Lemma 2. If φ is an *internal* formula, then we have

$$\mathsf{E\text{-}HA}^{\omega *} \vdash \forall r^{(\mathsf{d}^+\varphi)^*} \forall u^{\mathsf{d}^-\varphi} \left(|\varphi|^r_u \Leftrightarrow \varphi \right).$$

Lemma 3. Let Φ be a formula. For any $r, t : (\mathsf{d}^+\Phi)^*$ and $y : \mathsf{d}^-\Phi$, we have

$$\mathsf{E\text{-}HA}^{\omega *} \vdash |\Phi|^r_y \wedge r \preceq t \Rightarrow |\Phi|^t_y$$

where $r \preceq t$ means that r is contained in t, *i.e.* $\forall x (x \in r \Rightarrow x \in t)$.

Using the above two lemmas, we prove the soundness of the $\mathsf{D_{st}}$-interpretation.

Theorem 4 (Soundness)**.** Let Φ be a formula of system H and let Δ_{int} be a set of internal formulas. If

$$\mathsf{H} + \Delta_{\text{int}} \vdash \Phi$$

then from the proof we can extract a closed T^* term $r : (\mathsf{d}^+\Phi)^*$, called the *realiser* of Φ, such that

$$\mathsf{E\text{-}HA}^{\omega *} + \Delta_{\text{int}} \vdash \forall y^{\mathsf{d}^-\Phi} |\Phi|^r_y.$$

Proof. The proof is carried out by induction on the length of the derivation.
We firstly look at the case of the contraction axiom.

0. $A \Rightarrow A \wedge A$

   We define

$$\begin{aligned} U &:\equiv \lambda r.\ ((r_0, r_0) :: \cdots :: (r_{|r|-1}, r_{|r|-1}) :: []) \ : \ (\mathsf{d}^+A)^* \to (\mathsf{d}^+A \times \mathsf{d}^+A)^* \\ Y &:\equiv \lambda r. \lambda v.\ (v_0 :: v_1 :: []) \ : \ (\mathsf{d}^+A)^* \to (\mathsf{d}^-A \times \mathsf{d}^-A) \to (\mathsf{d}^-A)^*. \end{aligned}$$

   For any $(r, v) : (\mathsf{d}^+A)^* \times (\mathsf{d}^-A \times \mathsf{d}^-A)$, we have

$$|A \Rightarrow A \wedge A|^{[U,Y]}_{r,v} \quad = \quad \forall u \in (v_0 :: v_1 :: [])\, |A|^r_u \Rightarrow |A|^r_{v_0} \wedge |A|^r_{v_1}.$$

   In the usual Dialectica interpretation for HA^ω, decidability of primitive formulas (and hence of quantifier-free ones) is needed to realise the contraction axiom, because any given counterexample of $A \wedge A$ consists of two counterexamples of A and the decidability of $|A|^r_u$ (as it is quantifier-free) allows us to check which one is better. In the $\mathsf{D_{st}}$-interpretation, interpreted formulas $|A|^r_u$ may contain quantifiers and hence may not be decidable. However, decidability is not needed to realise the contraction axiom as shown above (or any axioms or rules).

We skip the other logical axioms and rules, and prove only the cases of non-standard axioms.

1. ($\forall^{\text{st}}$-intro) $\quad \forall x^\sigma (\mathsf{st}(x) \Rightarrow A(x)) \ \Rightarrow\ \forall^{\text{st}} x^\sigma A(x)$

   $\mathsf{d}^+(\forall x^\sigma(\mathsf{st}(x) \Rightarrow A(x))) = \sigma^* \to (\mathsf{d}^+A)^*$

   $\mathsf{d}^-(\forall x^\sigma(\mathsf{st}(x) \Rightarrow A(x))) = \sigma^* \times \mathsf{d}^-A$

   $\mathsf{d}^+(\forall^{\text{st}} x^\sigma A(x)) = \sigma \to (\mathsf{d}^+A)^*$

   $\mathsf{d}^-(\forall^{\text{st}} x^\sigma A(x)) = \sigma \times \mathsf{d}^-A$

   $\mathsf{d}^+(\forall^{\text{st}}\text{-intro}) = ((\sigma^* \to (\mathsf{d}^+A)^*)^* \to (\sigma \to (\mathsf{d}^+A)^*)^*) \times ((\sigma^* \to (\mathsf{d}^+A)^*)^* \to (\sigma \times \mathsf{d}^-A) \to (\sigma^* \times \mathsf{d}^-A)^*)$

   $\mathsf{d}^-(\forall^{\text{st}}\text{-intro}) = (\sigma^* \to (\mathsf{d}^+A)^*)^* \times (\sigma \times \mathsf{d}^-A)$

   We define

$$\begin{aligned} U &:\equiv \lambda F.[\lambda a.F[[a]]] \ : \ (\sigma^* \to (\mathsf{d}^+A)^*)^* \to (\sigma \to (\mathsf{d}^+A)^*)^* \\ Y &:\equiv \lambda F.\lambda(a,v).[([a],v)] \ : \ (\sigma^* \to (\mathsf{d}^+A)^*)^* \to (\sigma \times \mathsf{d}^-A) \to (\sigma^* \times \mathsf{d}^-A)^*. \end{aligned}$$

   For any $(F, (a, v)) : (\sigma^* \to (\mathsf{d}^+A)^*)^* \times (\sigma \times \mathsf{d}^-A)$, we have

$$\begin{aligned} |\forall^{\text{st}}\text{-intro}|^{[(U,Y)]}_{F,(a,v)} &= \forall y \in [([a], v)]\, \forall x^\sigma \left(x \in y_1 \Rightarrow |A(x)|^{F[y_1]}_{y_2} \right) \Rightarrow |A(a)|^{F[[a]]}_v \\ &\Leftrightarrow |A(a)|^{F[[a]]}_v \Rightarrow |A(a)|^{F[[a]]}_v. \end{aligned}$$

   The above realiser of $\forall^{\text{st}}$-intro (and similarly those of $\forall^{\text{st}}$-elim, $\exists^{\text{st}}$-intro, $\exists^{\text{st}}$-elim) is essentially the identity map. We only need to perform some sequence operations to make U and Y type-check.

2. ($\forall^{\mathrm{st}}$-elim) $\quad \forall^{\mathrm{st}} x^\sigma A(x) \;\Rightarrow\; \forall x^\sigma(\mathsf{st}(x) \Rightarrow A(x))$
$\mathsf{d}^+(\forall^{\mathrm{st}}\text{-elim}) = ((\sigma \to (\mathsf{d}^+A)^*)^* \to (\sigma^* \to (\mathsf{d}^+A)^*)^*) \times ((\sigma \to (\mathsf{d}^+A)^*)^* \to (\sigma^* \times \mathsf{d}^-A) \to (\sigma \times \mathsf{d}^-A)^*)$
$\mathsf{d}^-(\forall^{\mathrm{st}}\text{-elim}) = (\sigma \to (\mathsf{d}^+A)^*)^* \times (\sigma^* \times \mathsf{d}^-A)$
We define
$$U :\equiv \lambda F.[\lambda a.(F[a_0] +\!\!+ \cdots +\!\!+ F[a_{|a|-1}])] \;:\; (\sigma \to (\mathsf{d}^+A)^*)^* \to (\sigma^* \to (\mathsf{d}^+A)^*)^*$$
$$Y :\equiv \lambda F.\lambda(a,v).((a_0,v) :: \cdots :: (a_{|a|-1},v) :: []) \;:\; (\sigma \to (\mathsf{d}^+A)^*)^* \to (\sigma^* \times \mathsf{d}^-A) \to (\sigma \times \mathsf{d}^-A)^*$$
using the list recursor Rec^*, and have
$$\forall y \in Y(F,(a,v))\; y_2 = v \qquad \forall x \in a\; (x,v) \in Y(F,(a,v)).$$
For any $(F,(a,v)) : (\sigma \to (\mathsf{d}^+A)^*)^* \times (\sigma^* \times \mathsf{d}^-A)$, we have
$$\begin{aligned} |\forall^{\mathrm{st}}\text{-elim}|^{[(U,Y)]}_{F,(a,v)} &= \forall y \in Y(F,(a,v))\, |A(y_1)|^{F[y_1]}_{y_2} \Rightarrow \forall x \in a. |A(x)|^{F[a_0] +\!\!+ \cdots +\!\!+ F[a_{|a|-1}]}_v \\ &\Leftrightarrow \forall y \in Y(F,(a,v))\, |A(y_1)|^{F[y_1]}_{v} \Rightarrow \forall x \in a. |A(x)|^{F[a_0] +\!\!+ \cdots +\!\!+ F[a_{|a|-1}]}_v \end{aligned}$$
using Lemma 3 and the fact that $F[x] \preceq F[a_0] +\!\!+ \cdots +\!\!+ F[a_{|a|-1}]$ for all $x \in a$.

3. ($\exists^{\mathrm{st}}$-intro) $\quad \exists x^\sigma(\mathsf{st}(x) \wedge A(x)) \;\Rightarrow\; \exists^{\mathrm{st}} x^\sigma A(x)$
$\mathsf{d}^+(\exists x^\sigma(\mathsf{st}(x) \wedge A(x))) = \mathsf{d}^+(\exists^{\mathrm{st}} x^\sigma A(x)) = \sigma \times \mathsf{d}^+A$
$\mathsf{d}^-(\exists x^\sigma(\mathsf{st}(x) \wedge A(x))) = \mathsf{d}^-(\exists^{\mathrm{st}} x^\sigma A(x)) = (\mathsf{d}^-A)^*$
$\mathsf{d}^+(\exists^{\mathrm{st}}\text{-intro}) = ((\sigma \times \mathsf{d}^+A)^* \to (\sigma \times \mathsf{d}^+A)^*) \times ((\sigma \times \mathsf{d}^+A)^* \to (\mathsf{d}^-A)^* \to ((\mathsf{d}^-A)^*)^*)$
$\mathsf{d}^-(\exists^{\mathrm{st}}\text{-intro}) = (\sigma \times \mathsf{d}^+A)^* \times (\mathsf{d}^-A)^*$
We define
$$U :\equiv \lambda r.r \;:\; (\sigma \times \mathsf{d}^+A)^* \to (\sigma \times \mathsf{d}^+A)^*$$
$$Y :\equiv \lambda r.\lambda v.[v] \;:\; (\sigma \times \mathsf{d}^+A)^* \to (\mathsf{d}^-A)^* \to ((\mathsf{d}^-A)^*)^*.$$
For any $(r,v) : (\sigma \times \mathsf{d}^+A)^* \times (\mathsf{d}^-A)^*$, we have
$$\begin{aligned} |\exists^{\mathrm{st}}\text{-intro}|^{[(U,Y)]}_{r,v} &= \forall y \in [v] \exists x^\sigma \forall y' \in y \left(x \in r^1 \wedge |A(x)|^{r^2}_{y'}\right) \Rightarrow \exists x \in r^1 \forall y \in v\, |A(x)|^{r^2}_y \\ &\Leftrightarrow \exists x \in r^1 \forall y \in v\, |A(x)|^{r^2}_y \Rightarrow \exists x \in r^1 \forall y \in v\, |A(x)|^{r^2}_y. \end{aligned}$$

4. ($\exists^{\mathrm{st}}$-elim) $\quad \exists^{\mathrm{st}} x^\sigma A(x) \;\Rightarrow\; \exists x^\sigma(\mathsf{st}(x) \wedge A(x))$
The proof is almost the same as the one of ($\exists^{\mathrm{st}}$-intro).

5. ($\mathsf{IA}^{\mathsf{st}}$) We equivalently realise the rule ($\mathsf{IR}^{\mathsf{st}}$) of the external induction principle (as in [4]).
$$\frac{\Phi(0) \wedge \forall^{\mathrm{st}} n(\Phi(n) \Rightarrow \Phi(\mathsf{s}n))}{\forall^{\mathrm{st}} n\, \Phi(n)}\;(\mathsf{IR}^{\mathsf{st}})$$
Assume that we have a realiser of $\Phi(0)$, *i.e.* a term $r : (\mathsf{d}^+\Phi)^*$ such that
$$\forall u^{\mathsf{d}^-\Phi}\, |\Phi(0)|^r_u \tag{†}$$
and a realiser of $\forall^{\mathrm{st}} n(\Phi(n) \Rightarrow \Phi(\mathsf{s}n))$, *i.e.* a term $F : (\mathbb{N} \to (((\mathsf{d}^+\Phi)^* \to (\mathsf{d}^+\Phi)^*) \times ((\mathsf{d}^+\Phi)^* \to \mathsf{d}^-\Phi \to (\mathsf{d}^-\Phi)^*))^*)^*$ such that
$$\forall (n,t,v)^{\mathbb{N} \times (\mathsf{d}^+\Phi)^* \times \mathsf{d}^-\Phi} \left(\forall w \in F_2[n,t,v]\, |\Phi(n)|^t_w \Rightarrow |\Phi(\mathsf{s}n)|^{F_1[n,t]}_v\right) \tag{‡}$$
where $F_1 : (\mathbb{N} \to (\mathsf{d}^+\Phi)^* \to (\mathsf{d}^+\Phi)^*)^*$ and $F_2 : (\mathbb{N} \to (\mathsf{d}^+\Phi)^* t \to \mathsf{d}^-\Phi \to (\mathsf{d}^-\Phi)^*)^*$ are defined using F such that
$$F_1[n,t] = (F[n])^1[t] \qquad F_2[n,t,v] = (F[n])^2[t,v].$$
Then we define a term
$$T :\equiv \lambda n.\mathsf{Rec}(r, (\lambda m.\lambda x.F_1[m,x]), n) \;:\; \mathbb{N} \to (\mathsf{d}^+\Phi)^*$$
and have
$$T(0) = r \qquad T(\mathsf{s}n) = F_1[n, T(n)].$$
Now we show that $[T] : (\mathbb{N} \to (\mathsf{d}^+\Phi)^*)^*$ is a realiser of $\forall^{\mathrm{st}} n\, \Phi(n)$, *i.e.*
$$\forall (n,v)^{\mathbb{N} \times \mathsf{d}^-\Phi}\, |\Phi(n)|^{T(n)}_v$$

by induction on n (*i.e.* using the internal induction principle).

Base case. For any $u : \mathsf{d}^-\Phi$, we have $|\Phi(0)|_u^{T(0)} = |\Phi(0)|_u^r$ which holds due to (†).

Inductive step. Given n, assume $\forall v^{\mathsf{d}^-\Phi} |\Phi(n)|_v^{T(n)}$. Given $v : \mathsf{d}^-\Phi$, we have $|\Phi(\mathsf{s}n)|_v^{T(\mathsf{s}n)} = |\Phi(\mathsf{s}n)|_v^{F_1[n,T(n)]}$ which can be proved using (‡) and the induction hypothesis.

6. (I) $\forall^{\mathrm{st}} xs^{\sigma^*} \exists y^\tau \forall x \in xs\ \varphi(x,y) \Rightarrow \exists y^\tau \forall^{\mathrm{st}} x^\sigma \varphi(x,y)$

$\mathsf{d}^+(\forall^{\mathrm{st}} xs^{\sigma^*} \exists y^\tau \forall x \in xs\ \varphi(x,y)) = \sigma^* \to (\mathsf{d}^+\varphi)^*$

$\mathsf{d}^-(\forall^{\mathrm{st}} xs^{\sigma^*} \exists y^\tau \forall x \in xs\ \varphi(x,y)) = \sigma^* \times (\mathsf{d}^-\varphi)^*$

$\mathsf{d}^+(\exists y^\tau \forall^{\mathrm{st}} x^\sigma \varphi(x,y)) = \sigma \to (\mathsf{d}^+\varphi)^*$

$\mathsf{d}^-(\exists y^\tau \forall^{\mathrm{st}} x^\sigma \varphi(x,y)) = (\sigma \times \mathsf{d}^-\varphi)^*$

$\mathsf{d}^+(\mathsf{I}) = ((\sigma^* \to (\mathsf{d}^+\varphi)^*)^* \to (\sigma \to (\mathsf{d}^+\varphi)^*)^*) \times ((\sigma^* \to (\mathsf{d}^+\varphi)^*)^* \to (\sigma \times \mathsf{d}^-\varphi)^* \to (\sigma^* \times (\mathsf{d}^-\varphi)^*)^*)$

$\mathsf{d}^-(\mathsf{I}) = (\sigma^* \to (\mathsf{d}^+\varphi)^*)^* \times (\sigma \times \mathsf{d}^-\varphi)^*$

We define

$$U :\equiv \lambda r.\mathbf{c} \ : \ (\sigma^* \to (\mathsf{d}^+\varphi)^*)^* \to (\sigma \to (\mathsf{d}^+\varphi)^*)^*$$
$$Y :\equiv \lambda r.\lambda v.[(v^1, \mathbf{c})] \ : \ (\sigma^* \to (\mathsf{d}^+\varphi)^*)^* \to (\sigma \times \mathsf{d}^-\varphi)^* \to (\sigma^* \times (\mathsf{d}^-\varphi)^*)^*.$$

For any $(r,v) : (\sigma^* \to \mathsf{d}^+\varphi)^* \times (\sigma \times \mathsf{d}^-\varphi)^*$, we have

$$\begin{aligned} |\mathsf{I}|_{r,v}^{[(U,Y)]} &= \forall w \in [(v^1,\mathbf{c})] \exists y^\tau \forall u \in w_2 \forall x \in w_1 |\varphi(x,y)|_u^{r[w_1]} \Rightarrow \exists y^\tau \forall u \in v |\varphi(u_1,y)|_{u_2}^{\mathbf{c}} \\ &\Leftrightarrow \exists y^\tau \forall x \in v^1 \varphi(x,y) \Rightarrow \exists y^\tau \forall w \in v \varphi(w_1,y) \end{aligned}$$

by using Lemma 2 (φ is internal).

7. (NCR) $\forall y^\tau \exists^{\mathrm{st}} x^\sigma \Phi(x,y) \Rightarrow \exists^{\mathrm{st}} xs^{\sigma^*} \forall y^\tau \exists x \in_\sigma xs\, \Phi(x,y)$

$\mathsf{d}^+(\forall y^\tau \exists^{\mathrm{st}} x^\sigma \Phi(x,y)) = \sigma \times \mathsf{d}^+\Phi$

$\mathsf{d}^-(\forall y^\tau \exists^{\mathrm{st}} x^\sigma \Phi(x,y)) = (\mathsf{d}^-\Phi)^*$

$\mathsf{d}^+(\exists^{\mathrm{st}} xs^{\sigma^*} \forall y^\tau \exists x \in_\sigma xs\, \Phi(x,y)) = \sigma^* \times \mathsf{d}^+\Phi$

$\mathsf{d}^-(\exists^{\mathrm{st}} xs^{\sigma^*} \forall y^\tau \exists x \in_\sigma xs\, \Phi(x,y)) = ((\mathsf{d}^-\Phi)^*)^*$

$\mathsf{d}^+(\mathsf{NCR}) = ((\sigma \times \mathsf{d}^+\Phi)^* \to (\sigma^* \times \mathsf{d}^+\Phi)^*) \times ((\sigma \times \mathsf{d}^+\Phi)^* \to ((\mathsf{d}^-\Phi)^*)^* \to ((\mathsf{d}^-\Phi)^*)^*)$

$\mathsf{d}^-(\mathsf{NCR}) = (\sigma \times \mathsf{d}^+\Phi)^* \times ((\mathsf{d}^-\Phi)^*)^*$

We define

$$U :\equiv \lambda r.((r^1, r^2_0) :: (r^1, r^2_1) :: \cdots :: (r^1, r^2_{|r|-1}) :: []) \ : \ (\sigma \times \mathsf{d}^+\Phi)^* \to (\sigma^* \times \mathsf{d}^+\Phi)^*$$
$$Y :\equiv \lambda r.\lambda v.v \ : \ (\sigma \times \mathsf{d}^+\Phi)^* \to ((\mathsf{d}^-\Phi)^*)^* \to ((\mathsf{d}^-\Phi)^*)^*$$

where we write r^2_i for the ith element in the sequence $r^2 : (\mathsf{d}^+\Phi)^*$, and have

$$(U(r))^1 = [(r^1)^{|r|}] \qquad (U(r))^2 = r^2.$$

For any $(r,v) : (\sigma \times \mathsf{d}^+\Phi)^* \times ((\mathsf{d}^-\Phi)^*)^*$, we have

$$|\mathsf{NCR}|_{r,v}^{[(U,Y)]} = \forall u \in v \forall y^\tau \exists x \in r^1 \forall w \in u |\Phi(x,y)|_w^{r^2} \Rightarrow \exists xs \in [(r^1)^{|r|}] \forall u \in v \forall y^\tau \exists x \in xs \forall w \in u |\Phi(x,y)|_w^{r^2}.$$

8. (HAC) $\forall^{\mathrm{st}} x^\sigma \exists^{\mathrm{st}} y^\tau \Phi(x,y) \Rightarrow \exists^{\mathrm{st}} F^{\sigma \to \tau^*} \forall^{\mathrm{st}} x^\sigma \exists y \in F(x)\, \Phi(x,y)$

$\mathsf{d}^+(\forall^{\mathrm{st}} x^\sigma \exists^{\mathrm{st}} y^\tau \Phi(x,y)) = \sigma \to (\tau \times \mathsf{d}^+\Phi)^*$

$\mathsf{d}^-(\forall^{\mathrm{st}} x^\sigma \exists^{\mathrm{st}} y^\tau \Phi(x,y)) = \sigma \times (\mathsf{d}^-\Phi)^*$

$\mathsf{d}^+(\exists^{\mathrm{st}} F^{\sigma \to \tau^*} \forall^{\mathrm{st}} x^\sigma \exists y \in F(x)\, \Phi(x,y)) = (\sigma \to \tau^*) \times (\sigma \to (\mathsf{d}^+\Phi)^*)$

$\mathsf{d}^-(\exists^{\mathrm{st}} F^{\sigma \to \tau^*} \forall^{\mathrm{st}} x^\sigma \exists y \in F(x)\, \Phi(x,y)) = (\sigma \times (\mathsf{d}^-\Phi)^*)^*$

$\mathsf{d}^+(\mathsf{HAC}) = ((\sigma \to (\tau \times \mathsf{d}^+\Phi)^*)^* \to ((\sigma \to \tau^*) \times (\sigma \to (\mathsf{d}^+\Phi)^*))^*)$
$\quad \times ((\sigma \to (\tau \times \mathsf{d}^+\Phi)^*)^* \to (\sigma \times (\mathsf{d}^-\Phi)^*)^* \to (\sigma \times (\mathsf{d}^-\Phi)^*)^*)$

$\mathsf{d}^-(\mathsf{HAC}) = (\sigma \to (\tau \times \mathsf{d}^+\Phi)^*)^* \times (\sigma \times (\mathsf{d}^-\Phi)^*)^*$

We define

$$U :\equiv \lambda r.[(\lambda x.(r[x])^1, \lambda x.(r[x])^2)] \ : \ (\sigma \to (\tau \times \mathsf{d}^+\Phi)^*)^* \to ((\sigma \to \tau^*) \times (\sigma \to (\mathsf{d}^+\Phi)^*))^*$$

$$Y :\equiv \lambda r.\lambda v.v \ : \ (\sigma \to (\tau \times \mathsf{d}^+\Phi)^*)^* \to (\sigma \times (\mathsf{d}^-\Phi)^*)^* \to (\sigma \times (\mathsf{d}^-\Phi)^*)^*.$$

For any $(r, v) : (\sigma \to (\tau \times \mathsf{d}^+\Phi)^*)^* \times (\sigma \times (\mathsf{d}^-\Phi)^*)^*$, we have

$$\begin{aligned} |\mathsf{HAC}|^{[(U,Y)]}_{r,v} &= \forall u \in v \, \exists y \in (r[u_1])^1 \, \forall w \in u_2 \, |\Phi(u_1, y)|^{(r[u_1])^2}_w \\ &\qquad \Rightarrow \exists F \in [\lambda x.(r[x])^1] \, \forall u \in v \, \exists y \in F(u_1) \, \forall w \in u_2 \, |\Phi(u_1, y)|^{(r[u_1])^2}_w. \end{aligned}$$

9. $(\mathsf{HGMP}^{\mathsf{st}})$ $\quad (\forall^{\mathrm{st}} x^\sigma \varphi(x) \Rightarrow \psi) \ \Rightarrow \ \exists^{\mathrm{st}} xs^{\sigma^*} (\forall x \in xs \, \varphi(x) \Rightarrow \psi)$

   $\mathsf{d}^+(\forall^{\mathrm{st}} x^\sigma \varphi(x) \Rightarrow \psi) = ((\sigma \to (\mathsf{d}^+\varphi)^*)^* \to (\mathsf{d}^+\psi)^*) \times ((\sigma \to (\mathsf{d}^+\varphi)^*)^* \to \mathsf{d}^-\psi \to (\sigma \times \mathsf{d}^-\varphi)^*)$

   $\mathsf{d}^-(\forall^{\mathrm{st}} x^\sigma \varphi(x) \Rightarrow \psi) = (\sigma \to (\mathsf{d}^+\varphi)^*)^* \times \mathsf{d}^-\psi$

   $\mathsf{d}^+(\exists^{\mathrm{st}} xs^{\sigma^*} (\forall x \in xs \, \varphi(x) \Rightarrow \psi)) = \sigma^* \times (((\mathsf{d}^+\varphi)^* \to (\mathsf{d}^+\psi)^*) \times ((\mathsf{d}^+\varphi)^* \to \mathsf{d}^-\psi \to (\mathsf{d}^-\varphi)^*))$

   $\mathsf{d}^-(\exists^{\mathrm{st}} xs^{\sigma^*} (\forall x \in xs \, \varphi(x) \Rightarrow \psi)) = (\mathsf{d}^+\varphi \times \mathsf{d}^-\psi)^*$

   We define

   $$U :\equiv \lambda r.[((r^2[\mathbf{c}, \mathbf{c}])^1, \mathbf{c})] \ : \ (\mathsf{d}^+(\forall^{\mathrm{st}} x^\sigma \varphi(x) \Rightarrow \psi))^* \to (\mathsf{d}^+(\exists^{\mathrm{st}} xs^{\sigma^*} (\forall x \in xs \, \varphi(x) \Rightarrow \psi)))^*$$

   $$Y :\equiv \lambda r.\lambda v.[\mathbf{c}] \ : \ (\mathsf{d}^+(\forall^{\mathrm{st}} x^\sigma \varphi(x) \Rightarrow \psi))^* \to \mathsf{d}^-(\exists^{\mathrm{st}} xs^{\sigma^*} (\forall x \in xs \, \varphi(x) \Rightarrow \psi)) \to (\mathsf{d}^-(\forall^{\mathrm{st}} x^\sigma \varphi(x) \Rightarrow \psi))^*.$$

   For any $(r, v) : (\mathsf{d}^+(\forall^{\mathrm{st}} x^\sigma \varphi(x) \Rightarrow \psi))^* \times \mathsf{d}^-(\exists^{\mathrm{st}} xs^{\sigma^*} (\forall x \in xs \, \varphi(x) \Rightarrow \psi))$, we have

   $$\begin{aligned} |\mathsf{HGMP}^{\mathsf{st}}|^{[(U,Y)]}_{r,v} &= \forall u \in [\mathbf{c}] \left(\forall w \in r^2[u_1, u_2] \, \varphi(w_1) \Rightarrow \psi\right) \Rightarrow \exists xs \in [(r^2[\mathbf{c}, \mathbf{c}])^1] \, \forall u \in v \, (\forall x \in xs \, \varphi(x) \Rightarrow \psi) \\ &\Leftrightarrow (\forall w \in r^2[\mathbf{c}, \mathbf{c}] \, \varphi(w_1) \Rightarrow \psi) \Rightarrow (\forall x \in (r^2[\mathbf{c}, \mathbf{c}])^1 \, \varphi(x) \Rightarrow \psi) \end{aligned}$$

   by using Lemma 2 (both φ and ψ are internal) and the definition of canonical elements $\mathbf{c}^{\sigma \times \tau} :\equiv (\mathbf{c}^\sigma, \mathbf{c}^\tau)$.

10. $\mathsf{HIP}_{\forall^{\mathrm{st}}}$: $\ (\forall^{\mathrm{st}} x^\sigma \varphi(x) \Rightarrow \exists^{\mathrm{st}} y^\tau \Psi(y)) \ \Rightarrow \ \exists^{\mathrm{st}} ys^{\tau^*} (\forall^{\mathrm{st}} x^\sigma \varphi(x) \Rightarrow \exists y \in ys \, \Psi(y))$

    Let $A :\equiv \forall^{\mathrm{st}} x^\sigma \varphi(x) \Rightarrow \exists^{\mathrm{st}} y^\tau \Psi(y)$ and $B :\equiv \exists^{\mathrm{st}} ys^{\tau^*} (\forall^{\mathrm{st}} x^\sigma \varphi(x) \Rightarrow \exists y \in ys \, \Psi(y))$.

    $\mathsf{d}^+A = ((\sigma \to (\mathsf{d}^+\varphi)^*)^* \to (\tau \times \mathsf{d}^+\Psi)^*) \times ((\sigma \to (\mathsf{d}^+\varphi)^*)^* \to (\mathsf{d}^-\Psi)^* \to (\sigma \times \mathsf{d}^-\varphi)^*)$

    $\mathsf{d}^-A = (\sigma \to (\mathsf{d}^+\varphi)^*)^* \times (\mathsf{d}^-\Psi)^*$

    $\mathsf{d}^+B = \tau^* \times (((\sigma \to (\mathsf{d}^+\varphi)^*)^* \to (\mathsf{d}^+\Psi)^*) \times ((\sigma \to (\mathsf{d}^+\varphi)^*)^* \to (\mathsf{d}^-\Psi)^* \to (\sigma \times \mathsf{d}^-\varphi)^*))$

    $\mathsf{d}^-B = ((\sigma \to (\mathsf{d}^+\varphi)^*)^* \times (\mathsf{d}^-\Psi)^*)^*$

    We define

    $$U :\equiv \lambda r.[((r^1[\mathbf{c}])^1, (\lambda t.(r^1[\mathbf{c}])^2, \lambda t.\lambda v.r^2[\mathbf{c}, v]))] \ : \ (\mathsf{d}^+A)^* \to (\mathsf{d}^+B)^*$$

    $$Y :\equiv \lambda r.\lambda v.((\mathbf{c}, v^2_0) :: (\mathbf{c}, v^2_1) :: \cdots :: (\mathbf{c}, v^2_{|v|-1}) :: []) \ : \ (\mathsf{d}^+A)^* \to \mathsf{d}^-B \to (\mathsf{d}^-A)^*$$

    and have

    $$\forall u \in Y(r, v) \ u_1 = \mathbf{c} \qquad \forall u \in v \ (\mathbf{c}, u_2) \in Y(r, v).$$

    For any $(r, v) : (\mathsf{d}^+A)^* \times \mathsf{d}^-B$, we have

    $$\begin{aligned} |\mathsf{HIP}_{\forall^{\mathrm{st}}}|^{[(U,Y)]}_{r,v} &= \forall u \in Y(r, v) \left(\forall w \in r^2[u_1, u_2] \, \varphi(w_1) \Rightarrow \exists y \in (r^1[u_1])^1 \, \forall w \in u_2 \, |\Phi(y)|^{(r^1[u_1])^2}_w\right) \\ &\qquad \Rightarrow \exists ys \in [(r^1[\mathbf{c}])^1] \, \forall u \in v \left(\forall w \in r^2[\mathbf{c}, u_2] \, \varphi(w_1) \Rightarrow \forall w \in u_2 \, \exists y \in ys \, |\Phi(y)|^{(r^1[\mathbf{c}])^2}_w\right) \\ &\Leftrightarrow \forall u \in Y(r, v) \left(\forall w \in r^2[\mathbf{c}, u_2] \, \varphi(w_1) \Rightarrow \exists y \in (r^1[\mathbf{c}])^1 \, \forall w \in u_2 \, |\Phi(y)|^{(r^1[\mathbf{c}])^2}_w\right) \\ &\qquad \Rightarrow \forall u \in v \left(\forall w \in r^2[\mathbf{c}, u_2] \, \varphi(w_1) \Rightarrow \forall w \in u_2 \, \exists y \in (r^1[u_1])^1 \, |\Phi(y)|^{(r^1[\mathbf{c}])^2}_w\right) \end{aligned}$$

    by using Lemma 2 (φ is internal).

10′. $\mathsf{HIP}^{\mathrm{int}}_{\forall^{\mathrm{st}}}$: $\ (\forall^{\mathrm{st}} x^\sigma \varphi(x) \Rightarrow \exists^{\mathrm{st}} y^\tau \psi(y)) \ \Rightarrow \ \exists^{\mathrm{st}} ys^{\tau^*} (\forall^{\mathrm{st}} x^\sigma \varphi(x) \Rightarrow \exists y \in ys \, \psi(y))$

    We also consider a special case of $\mathsf{HIP}_{\forall^{\mathrm{st}}}$ where both φ, ψ are internal.

    Let $A :\equiv \forall^{\mathrm{st}} x^\sigma \varphi(x) \Rightarrow \exists^{\mathrm{st}} y^\tau \psi(y)$ and $B :\equiv \exists^{\mathrm{st}} ys^{\tau^*} (\forall^{\mathrm{st}} x^\sigma \varphi(x) \Rightarrow \exists y \in ys \, \psi(y))$.

    We define

    $$U :\equiv \lambda r.[((r^1[\mathbf{c}])^1, (\mathbf{c}, \lambda t.\lambda v.r^2[\mathbf{c}, \mathbf{c}]))] \ : \ (\mathsf{d}^+A)^* \to (\mathsf{d}^+B)^*$$

    $$Y :\equiv \lambda r.\lambda v.[\mathbf{c}] \ : \ (\mathsf{d}^+A)^* \to \mathsf{d}^-B \to (\mathsf{d}^-A)^*$$

For any $(r, v) : (\mathsf{d}^+ A)^* \times \mathsf{d}^- B$, we have

$$\begin{aligned} |\mathsf{HIP}^{\mathrm{int}}_{\forall^{\mathrm{st}}}|^{[(U,Y)]}_{r,v} \;&=\; \forall u \in [\mathbf{c}] \left(\forall w \in r^2[u_1, u_2]\, \varphi(w_1) \Rightarrow \exists y \in (r^1[u_1])^1\, \forall w \in u_2\, \psi(y)\right) \\ &\qquad \Rightarrow \exists ys \in [(r^1[\mathbf{c}])^1]\, \forall u \in v \left(\forall w \in r^2[\mathbf{c}, \mathbf{c}]\, \varphi(w_1) \Rightarrow \forall w \in u_2\, \exists y \in ys\, \psi(y)\right) \\ &\Leftrightarrow\; (\forall w \in r^2[\mathbf{c}, \mathbf{c}]\, \varphi(w_1) \Rightarrow \exists y \in (r^1[\mathbf{c}])^1\, \psi(y)) \\ &\qquad \Rightarrow \forall u \in v \left(\forall w \in r^2[\mathbf{c}, \mathbf{c}]\, \varphi(w_1) \Rightarrow \forall w \in u_2\, \exists y \in (r^1[\mathbf{c}])^1\, \psi(y)\right) \end{aligned}$$

by using Lemma 2 (both φ, ψ are internal).

□

Discussion. In order to develop the D_{st}-interpretation for system H in Agda, we figure out the types $\mathsf{d}^+\Phi$ of actual realisers and $\mathsf{d}^-\Phi$ of counterexamples of each formula Φ, and then inductively define an internal formula $|\Phi|^r_u$ for any formula Φ and terms $r : (\mathsf{d}^+\Phi)^*$ and $u : \mathsf{d}^-\Phi$. Strictly following our definition, we prove (the difficult and important part of) the soundness theorem. Our proof is less readable than the one in [4, §5] due to our formal, type-theoretic development, but makes the Agda implementation easier. The realisers of the nonstandard axioms in our soundness proof look different from those in [4, §5], but are essentially the same. Each nonstandard axiom is formulated as an implication $A \Rightarrow B$ and is realised by a pair (U, Y) of functions. Specifically U maps realisers of A to realisers of B, and Y maps counterexamples of B to counterexamples of A. Both U, Y are essentially the identity map, because realisers and counterexamples of A and B should contain the same computational information. In particular, realisers and counterexamples of internal formulas φ are non-computational. Therefore, to define simpler U, Y, if an input of type $\mathsf{d}^{+/-}\varphi$ (or equivalent to $\mathsf{d}^{+/-}\varphi$) is needed, we can choose the canonical element $\mathbf{c}$; and similarly we return $\mathbf{c}$ as an output of type $\mathsf{d}^{+/-}\varphi$ (or equivalent to $\mathsf{d}^{+/-}\varphi$).

3 References

[1] Edward Nelson, *Internal set theory: a new approach to nonstandard analysis*, Bulletin of the American Mathematical Society **83** (1977), no. 6, 1165–1198.

[2] Helmut Schwichtenberg and Stanley S. Wainer, *Proofs and computations*, Cambridge University Press, 2011.

[3] The Agda Development Team, *The Agda Wiki*. http://wiki.portal.chalmers.se/agda/pmwiki.php (accessed on 26-09-2016).

[4] Benno van den Berg, Eyvind Briseid, and Pavol Safarik, *A functional interpretation for nonstandard arithmetic*, Annals of pure and applied logic **163** (2012), no. 12, 1962–1994.