Simplicial Sets within Cubical Sets

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Computational Meaning of HoTT? (1)

HoTT is **Intensional Type Theory** together with Voevodsky's **Univalence Axiom** (UA) and **Higher Inductive Types** (HITs).

Let us recall what UA says. This requires a few definitions

$$\mathrm{iscontr}(X : \mathrm{Set}) = (\Sigma x : X)(\Pi y : X)\, \mathrm{Id}_X(x, y)$$

$$\begin{aligned}\mathrm{hfiber}(X, Y : \mathrm{Set})(f : X \to Y)(y : Y) &= \\ &= (\Sigma x : X)\, \mathrm{Id}_Y(f(x), y)\end{aligned}$$

$$\begin{aligned}\mathrm{isweq}(X, Y : \mathrm{Set})(f : X \to Y) &= \\ &= (\Pi y : Y)\, \mathrm{iscontr}(\mathrm{hfiber}(X, Y, f, y))\end{aligned}$$

$$\mathrm{Weq}(X, Y : \mathrm{Set}) = (\Sigma f : X \to Y)\, \mathrm{isweq}(X, Y, f)$$

Computational Meaning of HoTT? (2)

Using the eliminator J for identity types one easily defines a map

$$\mathrm{eqweq}(X, Y : \mathrm{Set}) : \mathrm{Id}_{\mathrm{Set}}(X, Y) \to \mathrm{Weq}(X, Y)$$

Then the **Univalence Axiom**

$$\mathsf{UA} : (\Pi X, Y : \mathrm{Set})\ \mathrm{isweq}(\mathrm{eqweq}(X, Y))$$

postulates that all maps $\mathrm{eqweq}(X, Y)$ are weak equivalences.

It has been shown that **simplicial sets** provide a model of HoTT interpreting types as Kan complexes.
But UA as it is **lacks computational meaning**:
what should be rewrite rules for the constant UA?

T. Coquand et.al. [CCHM] have developed a *Cubical Type Theory* with computational meaning in which one can **derive** UA.

From Simplicial Sets to Cubical Sets (1)

Bezem and Coquand have shown that the theory of simplicial sets is **not constructive**: one cannot even show constructively that Kan complexes are closed under exponentiation!

This limitation, however, can be overcome when working in **cubical sets**.

The crucial property is that representable objects are closed under finite products: in **sSet** the interval $\mathbb{I}$ is representable but $\mathbb{I}\times\mathbb{I}$ isn't!

The site of **sSet** is Δ, the full subcategory of **Poset** on finite non-empty linear posets.
The site of **cSet** is $\Box$, the full subcategory of **Poset** of finite powers of **2**.
Splitting idempotents in $\Box$ gives rise to **FL**, the full subcategory of **Poset** on finite lattices.

From Simplicial Sets to Cubical Sets (2)

Notice that $\square$ is op-equivalent to the category of *free finitely generated distributive lattices*. Coquand et.al. – just for convenience – used the opposite of the category of *free finitely generated de Morgan algebras*.

Let $i : \Delta \to \mathbf{FL}$ be the inclusion functor. The restriction functor i^* from $\mathbf{cSet} = \widehat{\mathbf{FL}}$ to $\mathbf{sSet} = \widehat{\Delta}$ has left and right adjoints $i_!$ and i_*, respectively. Since the restriction of the nerve functor Nv to **FL** is given by $i^* \circ \mathrm{Y}_{\mathbf{FL}}$ we have

$$i_*(X)(L) \cong \mathbf{cSet}(\mathrm{Y}_{\mathbf{FL}}(L), i_*(X)) \cong \mathbf{sSet}(i^*\mathrm{Y}_{\mathbf{FL}}(L)), X) \cong \mathbf{sSet}(\mathrm{Nv}(L), X)$$

from which it follows that i_* (and thus also $i_!$) is full and faithful. Since i^* has a left adjoint $i_!$ (given by left Kan extension of $\mathrm{Y}_{\mathbf{FL}} \circ i$ along Y_Δ) it preserves (finite) limits and thus $i^* \dashv i_*$ is an **injective geometric morphism**.

The topology on **FL** inducing **sSet**

A sieve $S \subseteq \mathrm{Y}_{\mathbf{FL}}(L)$ **covers** L iff $i^*S = i^*\mathrm{Y}_{\mathbf{FL}}(L) = \mathrm{Nv}(L)$ iff S contains all chains in L, i.e. all monotone maps $[n] \to L$.
There is a minimal covering sieve C_L consisting of all maps to L whose image is linearly ordered.

The corresponding **closure operator** $j : \Omega \to \Omega$ sends $S \subseteq \mathrm{Y}_{\mathbf{FL}}(L)$ to the set of all $u : K \to L$ with $uc \in S$ for all chains $c : [n] \to K$.

Cisinski Model Structures on **cSet** and **sSet** (1)

For Cisinski model structures on presheaf toposes $\widehat{\mathbb{C}}$ its class of **cofibrations** consists of all monos.
Its class of trivial fibrations consists of maps weakly right orthogonal to all monos.
A **naive fibration** is a map weakly right orthogonal to **all cylinders**, i.e. monos of the form

$$(\{\varepsilon\} \times X) \cup (\mathbb{I} \times Y) \hookrightarrow \mathbb{I} \times X$$

where $Y \subseteq X$, $\varepsilon \in \{0,1\} \subseteq \mathbb{I} = \mathrm{Y}(2)$.
The **trivial cofibrations** are the maps weakly left orthogonal to all naive fibrations.

On **sSet** this construction gives the classical model structure and on **cSet** the "type-theoretic" one of [CCHM].

Cisinski Model Structures on **cSet** and **sSet** (2)

Since i^* preserves $\mathbb{I}$ and finite limits and it retracts monos of **cSet** onto monos of **sSet** it follows that i^* retracts cylinders in **sSet** onto cylinders in **sSet**.
Thus $i_* p$ is a fibration in **cSet** iff p is a fibration in **sSet**.
Accordingly, we have $\mathcal{F}_{\mathbf{sSet}} = \mathbf{sSet} \cap \mathcal{F}_{\mathbf{cSet}}$.

Thus $i^* \dashv i_*$ is a *Quillen pair*.

We don't know whether it is a Quillen equivalence.
But presumably not!

Sattler's Theorem

The functor $i_!$ preserves monos, i.e. cofibrations, and also trivial cofibrations.

Thus $i_! \dashv i^*$ is also a Quillen pair.

But we also do not know whether it is a Quillen equivalence, i.e. whether all counits $\varepsilon_X : i_! i^* X \to X$ are weak equivalences w.r.t. the "type-theoretic" model structure on **cSet**.

i_* preserves and reflects "equality" of fibrant objects

Theorem If A and B are fibrant in **sSet**, i.e. Kan complexes, a map $f : A \to B$ is a weak equivalence in **sSet** iff it is a weak equivalence in **cSet**.

Proof.
One can show that $i_*(\mathrm{hfiber}(f)) \simeq \mathrm{hfiber}(i_* f)$ and thus
$\forall m \in \mathrm{Mono}(\mathbf{cSet})(m \perp \mathrm{hfiber}(i_* f))$ iff
$\forall m \in \mathrm{Mono}(\mathbf{cSet})(m \perp i_*(\mathrm{hfiber}(f)))$ iff
$\forall m \in \mathrm{Mono}(\mathbf{cSet})(i^* m \perp \mathrm{hfiber}(f))$ iff[1]
$\forall m \in \mathrm{Mono}(\mathbf{sSet})(m \perp \mathrm{hfiber}(f))$
i.e.[2] $i_* f$ is a weak equivalence in **cSet** iff f is a weak equivalence in **sSet**. □

[1] since the monos in **sSet** are precisely the sheafifications of monos in **cSet**

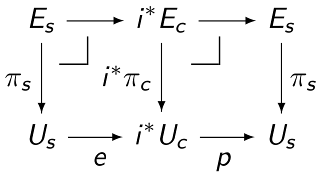
[2] as shown by Voevodsky for fibrant objects A and B a map $w : A \to B$ is a weak equivalence iff $\mathrm{hfiber}(w)$ is a trivial cofibrations, i.e. $m \perp \mathrm{hfiber}(w)$ for all monos m

Universes in **cSet** and **sSet** (1)

In both **cSet** and **sSet** we may construct universes

$$\pi_c : E_c \to U_c \quad \text{and} \quad \pi_s : E_s \to U_s$$

generic for small fibrations.
In both cases the universes are fibrant and univalent!
However, in case of **sSet** this requires heavy choice (due to use of minimal fibrations!) but not so for **cSet**.
Thus, we have



and $pe \sim \mathrm{id}_{U_s}$ (i.e. pe and id_{U_s} are homotopy equivalent) since the universe π_s is univalent.

Universes in **cSet** and **sSet** (2)

Since i_* preserves fibrations, pullbacks and $\sim$ for maps between fibrant objects we have

$$
\begin{array}{ccccc}
i_* E_s & \longrightarrow & i_* i^* E_c & \longrightarrow & i_* E_s \\
{\scriptstyle i_*\pi_s}\downarrow & \lrcorner & \downarrow{\scriptstyle i_* i^*\pi_c} & \lrcorner & \downarrow{\scriptstyle i_*\pi_s} \\
i_* U_s & \xrightarrow[i_* e]{} & i_* i^* U_c & \xrightarrow[i_* p]{} & i_* U_s
\end{array}
$$

with $i_* p \circ i_* e = i_*(p \circ e) \sim i_*(\mathrm{id}_{U_s}) = \mathrm{id}_{i_* U_s}$. Thus $i_* i^* \pi_c$ is generic for small fibrations which are families of sheaves and $i_* \pi_s$ is a univalent such universe.

Thus, pulling back $i_* i^* \pi_c$ along the homotopy equalizer of $i_* e \circ i_* p$ and $\mathrm{id}_{i_* i^* U_c}$ gives rise to a univalent universe U_{cs} in **cSet** which is weakly generic for small fibrations which are families of sheaves.

Conclusion and Outlook

- We have shown that simplicial sets form a(n essential) subtopos of cubical sets.
- Moreover, **sSet** is a submodel of **cSet** since the inclusion $i_* : \mathbf{sSet} \hookrightarrow \mathbf{cSet}$ preserves Σ and Π and also the interval $\mathbb{I}$ and thus also identity types.
- In **cSet** we have constructed a fibrant univalent universe generic for small fibrations of sheaves.

Thus, we conclude that

the simplical model lives within the cubical model!