Ordinal notation systems for ordinals below ε_0 in HoTT

Fredrik Nordvall Forsberg
University of Strathclyde, Glasgow

Joint work with Chuangjie Xu and Nicolai Kraus

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Ordinals, classically in set theory

Definition

A set α is an ordinal if it is transitive and $\in$ is well-founded on α:

- $x \in \alpha \to x \subseteq \alpha$,
- Every nonempty $X \subseteq \alpha$ has an $\in$-least element.

(Obviously too strong constructively!)

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E.g. already Turing [1949] used ordinals to prove termination of programs.

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- $\omega = \bigcup_{n \in \mathbb{N}} n$ is an ordinal.

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- $\omega \cdot 2 = \bigcup_{n < \omega} (\omega + n)$ is an ordinal.

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- $\omega^\omega = \bigcup_{n<\omega} \omega^n$ is an ordinal.
- ω^ω, ω^{ω^ω}, ... are ordinals;
- $\varepsilon_0 = \bigcup\{\omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ldots\}$ is an ordinal.

$\mathcal{E}_0$

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Fact (Cantor Normal Form)

Every ordinal α can be written uniquely as

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for some $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$.

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Hence if we compute the Cantor Normal Form

$$\beta_i = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_m}$$

and so on, we get decreasing sequences

$$\alpha > \beta_i > \gamma_j > \ldots$$

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and so on, we get decreasing sequences

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which must terminate. This gives a finite representation of α!

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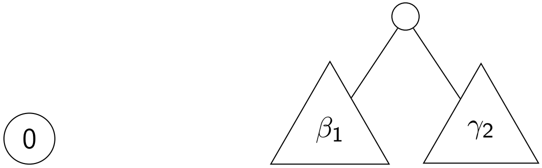
Simply binary trees!



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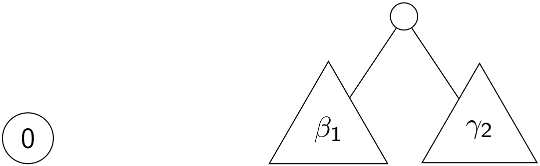

But: uniqueness of representation has been lost. How can we recover this?


A mutual approach

An inductive-inductive-recursive definition

We simultaneously define

$$\mathsf{data}\ \mathsf{MutualOrd} : \mathsf{Type}_0$$
$$\mathsf{data}\ \_<\_ : \mathsf{MutualOrd} \to \mathsf{MutualOrd} \to \mathsf{Type}_0$$
$$\mathsf{fst} : \mathsf{MutualOrd} \to \mathsf{MutualOrd}$$

MutualOrd

$$
\begin{array}{l}
\mathsf{data}\ \mathsf{MutualOrd}\ \mathsf{where} \\
\quad \mathbf{0} : \mathsf{MutualOrd} \\
\quad \omega\text{^}\_+\_[\_] : (a\ b : \mathsf{MutualOrd}) \to a \geq \mathsf{fst}\ b \to \mathsf{MutualOrd}
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MutualOrd

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data MutualOrd where
  0 : MutualOrd
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data _<_ where
  <₁ : 0 < ω^ a + b [ r ]
  <₂ : a < c → ω^ a + b [ r ] < ω^ c + d [ s ]
  <₃ : a ≡ c → b < d → ω^ a + b [ r ] < ω^ c + d [ s ]
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$$
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\textsf{data}\ \_<\_\ \textsf{where} \\
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\quad <_2 : a < c \to \omega\hat{}\ a + b\,[\,r\,] < \omega\hat{}\ c + d\,[\,s\,] \\
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Remark: there is an equivalent non-inductive-recursive definition where we define the graph of fst inductively.

Examples

- $\mathbf{1} = \omega\hat{}\ \mathbf{0} + \mathbf{0}\ [\ \mathsf{inj}_2\ \mathsf{refl}\ ]$
- $\omega = \omega\hat{}\ \mathbf{1} + \mathbf{0}\ [\ \mathsf{inj}_1\ <_1\ ]$
- $\omega\hat{}\langle\ a\ \rangle = \omega\hat{}\ a + \mathbf{0}\ [\ \geq\mathbf{0}\ ]$

Basic properties

Proposition

$\_<\_$ *is proof-irrelevant.*

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Proof.

We simultaneously define

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\begin{aligned}
&\mathsf{MutualOrdIsSet} : \mathsf{isSet}\ \mathsf{MutualOrd} \\
&\mathsf{<IsPropValued} : \mathsf{isProp}\ (a < b) \\
&\mathsf{MutualOrd}^{=} : \{r : a \geq \mathsf{fst}\ b\}\ \{s : c \geq \mathsf{fst}\ d\} \to a \equiv c \to b \equiv d \\
&\qquad\qquad \to \omega\hat{}\ a + b\ [\ r\ ] \equiv \omega\hat{}\ c + d\ [\ s\ ]
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$\square$

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\end{aligned}$$

□

Proposition

$\_<\_$ is trichotomous.

Proof.

We define

$$\mathsf{<\text{-}tri} : (a\ b : \mathsf{MutualOrd}) \to a < b \uplus a \geq b$$

using case distinctions on all subterms. □

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We now want to implement addition on MutualOrd. We simultaneously define

$$\begin{aligned} \_+\_ &: \mathsf{MutualOrd} \to \mathsf{MutualOrd} \to \mathsf{MutualOrd} \\ \geq\mathsf{fst}+ &: \{a : \mathsf{MutualOrd}\}\ (b\ c : \mathsf{MutualOrd}) \\ &\to a \geq \mathsf{fst}\ b \to a \geq \mathsf{fst}\ c \to a \geq \mathsf{fst}\ (b + c) \end{aligned}$$

Addition on MutualOrd

Remember: if $\alpha < \beta$ then $\omega^\alpha + \omega^\beta = \omega^\beta$.

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a + 0 = a
(ω^ a + c [ r ]) + (ω^ b + d [ s ]) with <-tri a b
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≥fst+ 0 _ r s = s
≥fst+ (ω^ _ + _ [ _ ]) 0 r s = r
≥fst+ (ω^ b + _ [ _ ]) (ω^ c + _ [ _ ]) r s with <-tri b c
... | inj₁ b<c = s
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Multiplication on MutualOrd

$$
\begin{aligned}
&\_\cdot\_ : \mathrm{MutualOrd} \to \mathrm{MutualOrd} \to \mathrm{MutualOrd} \\
&\mathbf{0} \cdot b = \mathbf{0} \\
&a \cdot \mathbf{0} = \mathbf{0} \\
&a \cdot (\omega\hat{}\ \mathbf{0} + d\,[\,r\,]) = a + a \cdot d \\
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\end{aligned}
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Note: All in terms of previous operations, so no simultaneous lemma needed.


A higher inductive approach

Uniqueness by making things the same

We want to avoid redundant representations of ordinals

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That is, we use a quotient inductive type.

A higher inductive appoach

```
data HITOrd : Type₀ where
  0 : HITOrd
  ω^_⊕_ : HITOrd → HITOrd → HITOrd
  swap : ∀ a b c → ω^ a ⊕ ω^ b ⊕ c ≡ ω^ b ⊕ ω^ a ⊕ c
  trunc : isSet HITOrd
```

(cf. finite multisets as a HIT [Licata, 2014]).

Example

```
example : (a b c : HITOrd)
  → ω^ a ⊕ ω^ b ⊕ ω^ c ⊕ 0 ≡ ω^ c ⊕ ω^ b ⊕ ω^ a ⊕ 0
example a b c = begin
  ω^ a ⊕ ω^ b ⊕ ω^ c ⊕ 0 ≡⟨ swap a b _ ⟩
  ω^ b ⊕ ω^ a ⊕ ω^ c ⊕ 0 ≡⟨ cong (ω^ b ⊕_) (swap a c _) ⟩
  ω^ b ⊕ ω^ c ⊕ ω^ a ⊕ 0 ≡⟨ swap b c _ ⟩
  ω^ c ⊕ ω^ b ⊕ ω^ a ⊕ 0 □
```

Arithmetic on HITOrd

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For arithmetic, these are the so-called Hessenberg sum and product [Hessenberg, 1906].

Hessenberg sum

```
_⊕_ : HITOrd → HITOrd → HITOrd
0                ⊕ y = y
(ω^ a ⊕ b)       ⊕ y = ω^ a ⊕ (b ⊕ y)
(swap a b c i)   ⊕ y = swap a b (c ⊕ y) i
(trunc p q i j)  ⊕ y = trunc (cong (_⊕ y) p) (cong (_⊕ y) q) i j
```

Hessenberg sum

$$
\begin{array}{l}
\_\oplus\_ : \mathsf{HITOrd} \to \mathsf{HITOrd} \to \mathsf{HITOrd} \\
\mathbf{0} \qquad\qquad\qquad\quad \oplus\ y = y \\
(\omega\hat{}\ a \oplus b) \quad\ \ \oplus\ y = \omega\hat{}\ a \oplus (b \oplus y) \\
(\mathsf{swap}\ a\ b\ c\ i) \oplus y = \mathsf{swap}\ a\ b\ (c \oplus y)\ i \\
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In the swap case, we have to construct a path

$$\omega\hat{}\ a \oplus \omega\hat{}\ b \oplus (c \oplus y) \equiv \omega\hat{}\ b \oplus \omega\hat{}\ a \oplus (c \oplus y)$$

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Proposition

$\_\oplus\_$ *is commutative.*

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Theorem

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Corollary

MutualOrd and HITOrd are identical, i.e. there is a proof $M{\equiv}H$: $MutualOrd \equiv HITOrd$.

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$$\begin{tikzcd}
\mathsf{MutualOrd} \arrow[r, bend right] & \mathsf{HITOrd} \arrow[l, bend right]
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By using univalence, we can transport operations and proofs between MutualOrd and HITOrd.

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Transporting proofs

We can also transport properties. For instance: define

$$\mathsf{Dec} : (A : \mathsf{Type}\ \ell) \to (A \to A \to \mathsf{Type}\ \ell') \to \mathsf{Type}\ (\ell \sqcup \ell')$$
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Hence we can construct

$$\mathsf{<^{H}\text{-}dec} : \mathsf{Dec}\ \mathsf{HITOrd}\ \_{<^{H}}\_$$
$$\mathsf{<^{H}\text{-}dec} = \mathsf{transport}\ (\lambda\ i \to \mathsf{Dec}\ (\mathsf{M{\equiv}H}\ i)\ (\mathsf{<Path}\ i))\ \mathsf{<\text{-}dec}$$

where

$$\mathsf{<Path} : \mathsf{PathP}\ (\lambda\ i \to \mathsf{M{\equiv}H}\ i \to \mathsf{M{\equiv}H}\ i \to \mathsf{Type}_0)\ \_{<}\_\ \_{<^{H}}\_$$

is a dependent path from $\_{<}\_$ to $\_{<^{H}}\_$.

It computes!

Define

$$\mathsf{lt} : \mathsf{HITOrd} \to \mathsf{HITOrd} \to \mathsf{Bool}$$
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$$\begin{aligned}
&\text{Ex}[{<^{\text{H}}}\text{-decComp}] : \\
&\qquad \text{lt}\ \mathbf{0}\ \mathbf{0} \equiv \text{false} \\
&\quad \times\ \text{lt}\ \text{H}.\omega\ ((\text{H}.\mathbf{1} \oplus \text{H}.\mathbf{1}) \otimes \text{H}.\omega) \equiv \text{true} \\
&\quad \times\ \text{lt}\ (\text{H}.\omega\text{\textasciicircum}\langle\ \text{H}.\omega\ \rangle)\ (\text{H}.\omega\text{\textasciicircum}\langle\ \text{H}.\mathbf{1} +^{\text{H}} \text{H}.\omega\ \rangle) \equiv \text{false} \\
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\end{aligned}$$

$$\mathsf{Ex}[\oplus^{\mathsf{M}}\mathsf{Comp}] : \mathsf{M}.\mathbf{1} \oplus^{\mathsf{M}} \mathsf{M}.\omega \equiv \mathsf{M}.\omega + \mathsf{M}.\mathbf{1}$$
$$\mathsf{Ex}[\oplus^{\mathsf{M}}\mathsf{Comp}] = \mathsf{refl}$$

Summary and outlook

Conclusions

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Thanks!