Categories with Families

Unityped, Simply Typed, Dependently Typed

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Vladimir Voevodsky:

A type system is not a mathematical notion.

- So what is the mathematical notion corresponding to a type system?
- It should be a **generalized algebraic theory**.
- Or perhaps an **essentially algebraic theory**.

The generalized algebraic theory of cwfs

- is closely related to Martin-Löf's 1992 **substitution calculus** for dependent type theory (which gives rise to a free cwf);
- can be structured using categorical concepts.

When connecting syntactic type systems and categorical notions, cwfs are in the middle:

- Syntactic type systems form free cwfs (with extra structure)
- and suitable more mainstream categorical notions can be proved to be equivalent, in a sense to be made precise.

The essentially algebraic theory of natural models

- is closely related to Martin-Löf’s 1992 **substitution calculus** for dependent type theory;
- can be structured using categorical concepts (representable natural transformation of presheaves).

See Newstead PhD "Algebraic models of dependent type theory" 2018

Ucwfs, scwfs, cwfs

All are based on a category of contexts $\mathcal{C}$ with a terminal object.

Ucwfs have only one type and one presheaf of terms

$$\mathrm{Tm} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$$

Scwfs have a set of types and one presheaf of terms

$$\mathrm{Tm}_A : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$$

for each type A.

Cwfs have a functor

$$T : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fam}$$

or equivalently, two presheaves

$$\begin{aligned} \mathrm{Ty} &: \mathcal{C}^{\mathrm{op}} \to \mathrm{Set} \\ \mathrm{Tm} &: \left(\int^{\mathcal{C}} \mathrm{Ty}\right)^{\mathrm{op}} \to \mathrm{Set} \end{aligned}$$

Context comprehension

Ucwfs assign for each $n \in \mathcal{C}_0$ a representation $\mathrm{s}(n) \in \mathcal{C}_0$ of the presheaf

$$\mathcal{C}(-,n) \times \mathrm{Tm}(-) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$$

Scwfs assign for each $\Gamma \in \mathcal{C}_0$ and $A \in \mathrm{Ty}$ a representation $\Gamma.A \in \mathcal{C}_0$ of the presheaf

$$\mathcal{C}(-,\Gamma) \times \mathrm{Tm}_A(-) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$$

Cwfs assign for each $\Gamma \in \mathcal{C}_0$ and $A \in \mathrm{Ty}(\Gamma)$ a representation $\Gamma.A \in \mathcal{C}_0$ of the presheaf

$$\sum_{\gamma \in \mathcal{C}(-,\Gamma)} \mathrm{Tm}(-, A[\gamma]) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$$

Contextuality

A cwf is **contextual** iff there is a length function

$$l : \mathcal{C}_0 \to \mathbb{N}$$

such that

- $l(\Gamma) = 0$ iff $\Gamma = 1$ and
- $l(\Gamma) = n + 1$ iff there are unique Δ and A such that $\Gamma = \Delta.A$ and $l(\Delta) = n$.

Cf Cartmell's 1978 **contextual categories**. Note that unlike the other parts of the definition of cwfs

- it does not correspond to an inference rule of dependent type theory;
- it is not expressed in the language of generalized algebraic theories;
- however, free cwfs are contextual.

Contextual ucwfs

Equivalences of categories (structure strictly preserved):

- with cartesian operads (one object multicategories):

$$\mathbf{CartOperad} \rightleftarrows \mathbf{Ucwf}_{\mathrm{ctx}}$$

- with Lawvere theories:

$$\mathbf{LawTh} \rightleftarrows \mathbf{Ucwf}_{\mathrm{ctx}}$$

We have similar equivalences between **contextual scwfs**, coloured operads (multicategories), and many-sorted Lawvere theories.

Predicate logic

Predicate logic is modelled by **ucwf-indexed scwfs** $(\mathcal{C}, \mathrm{Tm}, \mathcal{P})$ with extra structure for the logical constants.

- $(\mathcal{C}, \mathrm{Tm})$ is a ucwf;
- $\mathcal{P} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Scwf}$ is a functor into the category of scwfs and strict scwf-morphisms.
  - $\mathcal{P}(n)$ is the scwf of propositions and proofs in n term variables.
  - If $\gamma \in \mathcal{C}(n,m)$, then $\mathcal{P}(\gamma) : \mathcal{P}(m) \to \mathcal{P}(n)$ is the strict scwf-morphism which applies the substitution γ to the different components of the scwf $\mathcal{P}(m)$.

These are similar to Lawvere's **hyperdoctrines** (which are based on indexed categories), but closer to the usual formal system of predicate logic. Note that they yield a generalized algebraic theory for predicate logic.

Untyped lambda calculus

A $\lambda\beta\eta$-structure on a ucwf $(\mathcal{C}, \mathrm{Tm})$ is a natural isomorphism of presheaves

$$\mathrm{Tm}(\mathrm{s}(-)) \overset{\lambda}{\cong} \mathrm{Tm}(-)$$

where the functorial action of s is defined as $\mathrm{s}(\gamma) = \langle \gamma \circ \mathrm{p}, \mathrm{q} \rangle$.
We have the following equivalence:

$$\mathbf{LawTh}^{\lambda\beta\eta} \rightleftarrows \mathbf{Ucwf}^{\lambda\beta\eta}_{\mathrm{ctx}}$$

where $\mathbf{LawTh}^{\lambda\beta\eta}$ are Obtułowicz' 1977 algebraic theories of type $\lambda\beta\eta$.

Some type structures

A $\times$**-type structure** on an scwf $(\mathcal{C}, \mathrm{Ty}, \mathrm{Tm})$ assigns to each type A and B a representing type $A \times B$ and a natural isomorphism of preheaves

$$\mathrm{Tm}(-, A) \times \mathrm{Tm}(-, B) \overset{\langle -,- \rangle}{\cong} \mathrm{Tm}(-, A \times B)$$

A Σ**-type structure** on a cwf $(\mathcal{C}, \mathrm{Ty}, \mathrm{Tm})$ consists of a natural transformation Σ of type presheaves

$$\sum_{A \in \mathrm{Ty}(-)} \mathrm{Ty}(- \cdot A) \to \mathrm{Ty}(-)$$

and for each $A \in \mathrm{Ty}(\Gamma)$ and $B \in \mathrm{Ty}(\Gamma \cdot A)$ an isomorphism

$$\sum_{a \in \mathrm{Tm}(\Gamma, A)} \mathrm{Tm}(\Gamma, B[\langle \mathrm{id}, a \rangle]) \overset{\langle -,- \rangle}{\cong} \mathrm{Tm}(\Gamma, \Sigma_\Gamma(A, B))$$

which is stable under substitution.

An equivalence of categories

Let

- $\mathbf{CCs}$ be the category of small cartesian categories (finite products as structure) and functors which preserve the finite products strictly.
- $\mathbf{Scwf}^{\mathbb{N}_1,\times}_{\mathrm{ctx}}$ be the category of small contextual scwfs with finite product types and strict scwf-morphisms preserving finite product types strictly.

Then

$$\mathbf{CCs} \underset{\mathbf{C}}{\overset{\mathbf{L}}{\rightleftarrows}} \mathbf{Scwf}^{\mathbb{N}_1,\times}_{\mathrm{ctx}}$$

We also have (cf Lambek and Scott)

$$\mathbf{CCCs} \underset{\mathbf{C}}{\overset{\mathbf{L}}{\rightleftarrows}} \mathbf{Scwf}^{\mathbb{N}_1,\times,\Rightarrow}_{\mathrm{ctx}}$$

An equivalence of categories

$$\mathbf{CCs} \underset{\mathbf{C}}{\overset{\mathbf{L}}{\rightleftarrows}} \mathbf{Scwf}_{\text{ctx}}^{\mathbb{N}_1,\times}$$

- The functor $\mathbf{L}$ maps a cartesian category $\mathcal{D}$ to a contextual scwf with product types $(\text{List}(\mathcal{D}), \mathcal{D}_0, \text{Tm})$ where $\text{Tm}([A_1, A_2, \ldots, A_n], A) = \mathcal{D}((\cdots(A_1 \times A_2) \times \cdots) \times A_n, A)$, etc.
- The functor $\mathbf{C}$ maps a contextual scwf with product types $(\mathcal{C}, \text{Ty}, \text{Tm})$ to the cartesian category where objects are types, and arrows from A to B are terms in $\text{Tm}(1.A, B)$.
- We have $\mathbf{C}(\mathbf{L}(\mathcal{D})) = \mathcal{D}$.
- We have $\mathbf{L}(\mathbf{C}(\mathcal{C}, \text{Ty}, \text{Tm})) = (\text{List}(\mathbf{C}(\mathcal{C}, \text{Ty}, \text{Tm})), \text{Ty}, \text{Tm}') \cong (\mathcal{C}, \text{Ty}, \text{Tm})$.

An equivalence of types in univalent type theory

Let

- $\mathbf{CCs}$ be the **(metalanguage) type** of small **strict** cartesian categories with finite products as structure. Such categories have hsets of objects and arrows.
- $\mathbf{Scwf}_{\mathrm{ctx}}^{\mathbb{N}_1,\times}$ be the **(metalanguage) type** of small **strict** contextual scwfs with finite product (object language) types. They have hsets of objects, arrows, types, and terms.

Then

$$\mathbf{CCs} \underset{\mathbf{C}}{\overset{\mathbf{L}}{\rightleftarrows}} \mathbf{Scwf}_{\mathrm{ctx}}^{\mathbb{N}_1,\times}$$

is an equivalence of types. Cf Ahrens, Lumsdaine, Voevodsky 2017 "Categorical structures for type theory in univalent foundations".

An equivalence of types in univalent type theory

$$\mathbf{CCs} \underset{\mathbf{C}}{\overset{\mathbf{L}}{\rightleftarrows}} \mathbf{Scwf}^{\mathbb{N}_1,\times}_{\mathrm{ctx}}$$

- $\mathbf{L}$ and $\mathbf{C}$ are **functions**, not functors.
- We have $\mathbf{L}(\mathbf{C}(\mathcal{C}, \mathrm{Ty}, \mathrm{Tm})) = (\mathcal{C}, \mathrm{Ty}, \mathrm{Tm})$ by the univalence axiom and the structure identity principle.
- In the **groupoid model** the equivalence becomes an equivalence of groupoids.

Two theorems

P. Clairambault, PD, TLCA 2011, MSCS 2014:

- A biequivalence of 2-categories

$$\mathbf{LCC} \simeq \mathbf{Cwf}_{\mathrm{dem}}^{\mathrm{I}_{\mathrm{ext}},\Sigma,\Pi}$$

- Another biequivalence of 2-categories

$$\mathbf{FL} \simeq \mathbf{Cwf}_{\mathrm{dem}}^{\mathrm{I}_{\mathrm{ext}},\Sigma}$$

Prove them in univalent type theory!

Democracy

A cwf is **democratic** provided each context Γ is represented by a type $\overline{\Gamma}$ such that there is an isomorphism

$$\mathrm{d}_{\Gamma} : \Gamma \cong 1.\overline{\Gamma}$$

- it does not correspond to an inference rule of dependent type theory;
- however, the free cwf is democratic;
- democracy can be expressed in the language of generalized algebraic theories.

A biequivalence of 2-categories

Let

- $\mathbf{CCp}^2$ be the 2-category of small cartesian categories (finite products as property), functors which preserve this property, and natural transformations.
- $\mathbf{Scwf}^2_{\mathrm{dem}}$ be the 2-category of small democratic scwfs with finite product types, pseudo scwf-morphisms preserving finite product types up to isomorphism, and natural transformations.

$$\mathbf{CCp}^2 \underset{\mathbf{C}^2}{\overset{\mathbf{L}^2}{\rightleftarrows}} \mathbf{Scwf}^2_{\mathrm{dem}}$$

A biequivalence of 2-categories

$$\mathbf{CCp}^2 \underset{\mathbf{C}^2}{\overset{\mathbf{L}^2}{\rightleftarrows}} \mathbf{Scwf}^2_{\mathrm{dem}}$$

- $\mathbf{L}^2$ is the 2-functor that maps a cartesian category $\mathcal{D}$ to a democratic scwf $(\mathcal{D}, \mathrm{Ty}, \mathrm{Tm})$ where $\mathrm{Ty} = \mathcal{D}_0$ and $\mathrm{Tm}(\Gamma, A) = \mathcal{D}(\Gamma, A)$.
- $\mathbf{C}^2$ is the forgetful 2-functor that maps a democratic scwf $(\mathcal{C}, \mathrm{Ty}, \mathrm{Tm})$ to its cartesian category of contexts $\mathcal{C}$. We have $\Gamma \times_{\mathbf{CCp}^2} \Delta = \Gamma.\overline{\Delta}$, etc.
- We have $\mathbf{C}^2(\mathbf{L}^2(\mathcal{D})) = \mathcal{D}$
- and the equivalence $\mathbf{L}^2(\mathbf{C}^2(\mathcal{C}, \mathrm{Ty}, \mathrm{Tm})) = (\mathcal{C}, \mathcal{C}_0, \mathrm{Tm}') \simeq (\mathcal{C}, \mathrm{Ty}, \mathrm{Tm})$ in $\mathbf{Scwf}^2_{\mathrm{dem}}$

The equivalence in $\mathbf{Scwf}^2_{\mathrm{dem}}$

We define 2-natural transformations of 2-functors

$$\mathrm{I}_{\mathbf{Scwf}^2_{\mathrm{dem}}} \underset{\varepsilon}{\overset{\eta}{\rightleftarrows}} \mathbf{L}^2\mathbf{C}^2$$

which are inverses up to invertible modifications. For each scwf $(\mathcal{C}, \mathrm{Ty}, \mathrm{Tm})$ we construct pseudo cwf-morphisms consisiting of

- the identity functors in both directions on $\mathcal{C}$;
- on types:

$$\eta^{\mathrm{Ty}}(A) = 1.A \qquad \varepsilon^{\mathrm{Ty}}(\Gamma) = \overline{\Gamma}$$

- on terms:

$$\eta^{\mathrm{Tm}}{}_{\Gamma,A}(a) = \langle\langle\rangle, a\rangle \qquad \varepsilon^{\mathrm{Tm}}{}_{\Delta,\Gamma}(\gamma) = \mathrm{q}_{1,\overline{\Gamma}}[\mathrm{d}_\Gamma \circ \gamma]$$

Summary

- Generalized algebraic theories of the untyped lambda calculus, typed lambda calculus, predicate logic, and dependent type theory.
- Correspondence theorems with basic categorical notions using contextuality.
- Weak notions of model: property vs structure. Correspondence theorems using democracy.
- Rewrite Lambek and Scott based on ucwfs, scwfs, cwfs and leading up to the biequivalence with lcccs. See Castellan, Clairambault, Dybjer 2019, arXiv:1904.00827.
- Rewrite Lambek and Scott in univalent type theory!