Composition, Filling, and Fibrancy of the Universe

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QMS and models of HoTT

A model of HoTT should be a Quillen model structure with some special properties.

One way to construct such a model in a topos $\mathcal{E}$ is via a *premodel*:

Definition

A *premodel* in a topos $\mathcal{E}$ consists of $(\Phi, \mathbb{I}, \mathsf{V})$ where:

- Φ is a representable class of monos $\Phi \hookrightarrow \Omega$, such that ...
- $\mathbb{I}$ is an interval $1 \rightrightarrows \mathbb{I}$, such that ...
- $\dot{\mathsf{V}} \to \mathsf{V}$ is a universe of *small families*, such that ...

Such a set-up was used by Orton-Pitts to construct models of HoTT in the *extensional* type theory of $\mathcal{E}$, but it can also be used to construct a QMS, as was originally done by Sattler.

QMS from a premodel

Along the way, one needs to construct a universe $\dot{\mathsf{U}} \to \mathsf{U}$ of fibrant types and show that U is itself fibrant.

There are four man steps:

1. show that Kan filling can be reduced to composition,
2. prove the equivalence extension property for fibrations,
3. construct the universe of fibrations, which is univalent by (2),
4. show that a univalent universe has composition, and is thus Kan by (1).

We will sketch (1), (3), and (4) today.
We first recall some basic definitions.

1. The cofibration awfs ($\mathcal{C}$, TFib)

The monos $C \rightarrowtail Z$ classified by $\Phi \hookrightarrow \Omega$ are the *cofibrations* $\mathcal{C}$.

$$
\begin{tikzcd}
C \arrow[rr] \arrow[dr, dotted] \arrow[dd, tail] & & 1 \arrow[dd] \\
& 1 \arrow[ur] \arrow[dd, tail] & \\
Z \arrow[rr] \arrow[dr, dotted] & & \Omega \\
& \Phi \arrow[ur, hook] &
\end{tikzcd}
$$

These are closed under pullbacks.

The cofibration awfs $(\mathcal{C}, \mathsf{TFib})$

The generic cofibration $1 \rightarrowtail \Phi$ determines a polynomial endofunctor,

$$X^+ := \sum_{\varphi:\Phi} X^\varphi .$$

This is a (fibered) monad,

$$+ : \mathcal{E} \longrightarrow \mathcal{E}$$

because of ...

The cofibration awfs $(\mathcal{C}, \mathsf{TFib})$

In each slice $\mathcal{E}/X$, the algebras (A, α) for the underlying pointed endofunctor,

$$+_X : \mathcal{E}/X \longrightarrow \mathcal{E}/X$$

are the *trivial fibrations*.

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & A^+ \arrow[ll, bend right, dotted, "\alpha"'] \arrow[dl] \\ & X & \end{tikzcd}$$

They form the right class of the *cofibration awfs* $(\mathcal{C}, \mathsf{TFib})$.

The cofibration awfs $(\mathcal{C}, \mathsf{TFib})$

The algebra structures on a trivial fibration correspond uniquely to *uniform right lifting structures* against the cofibrations,

$$\begin{tikzcd} C \arrow[r] \arrow[d, tail] & A \arrow[d, "\shortmid" marking] \\ Z \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

that is, a coherent choice of diagonal fillers.

The fibration awfs $(\mathsf{TCof}, \mathcal{F})$

For any map $u : A \to B$ in $\mathcal{E}$, the *Leibniz adjunction*

$$(-)\otimes u \;\dashv\; u\Rightarrow(-)$$

relates the pushout-product with u and the pullback-hom with u.

The functors $(-\otimes u) \dashv (u\Rightarrow -) : \mathcal{E}^{\mathbb{2}} \longrightarrow \mathcal{E}^{\mathbb{2}}$ also satisfy

$$(c\otimes u) \boxslash f \;\Leftrightarrow\; c \boxslash (u\Rightarrow f)$$

with respect to the diagonal filling relation $c \boxslash f$.

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "c\otimes u"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] \arrow[ur, dotted] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r] \arrow[d, "c"'] & \cdot \arrow[d, "u\Rightarrow f"] \\ \cdot \arrow[r] \arrow[ur, dotted] & \cdot \end{tikzcd}$$

The fibration awfs $(\mathsf{TCof}, \mathcal{F})$

Let us assume that the interval $\mathbb{I}$ has *connections*,

$$\wedge, \vee : \mathbb{I} \times \mathbb{I} \longrightarrow \mathbb{I}\,,$$

making it a distributive lattice. This holds e.g. for $\mathcal{E} = \mathsf{Set}^{\mathbb{C}^{op}}$, presheaves on the *Dedekind cubes* $\mathbb{C}$, the full subcategory of **Cat** on the finite powers of $\mathbb{2}$ (the Lawvere theory of distributive lattices).

We then define the fibrations in terms of the trivial fibrations by:

$$f \in \mathcal{F} \quad \text{iff} \quad (\delta \Rightarrow f) \in \mathsf{TFib}$$

using the pullback-hom $\delta \Rightarrow f$ with *both* endpoints $\delta : 1 \to \mathbb{I}$.

The fibration awfs $(\mathsf{TCof}, \mathcal{F})$

Definition

A map $f : Y \to X$ is a *fibration* if $\delta \Rightarrow f$ is a trivial fibration. Equivalently by transposition, $f \in \mathcal{F}$ iff

$$c \otimes \delta \boxslash f ,$$

$$\begin{tikzcd}
Z +_C (C \times \mathbb{I}) \arrow[r] \arrow[d, tail, "c\otimes\delta"'] & Y \arrow[d, "f"] \\
Z \times \mathbb{I} \arrow[r] \arrow[ur, dotted] & X
\end{tikzcd}$$

for all cofibrations $c : C \rightarrowtail Z$.

(This may be called "partial open box filling")

Filling and Composition

Definition

Say that an object X *has filling* if it is fibrant in this sense, i.e. for all cofibrations $c : C \rightarrowtail Z$, there is a diagonal filler as in

$$\begin{tikzcd} C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathbb{I}} \arrow[d, "\delta \Rightarrow X"] \\ Z \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

where the Leibniz exponential $\delta \Rightarrow X : X^{\mathbb{I}} \to X$ is "evaluation at the endpoint $\delta : 1 \to \mathbb{I}$" (and we require the condition for both endpoints).

Filling and Composition

Definition

Say that X *has composition* if for all cofibrations $c : C \rightarrowtail Z$, there is a diagonal filler k making both subdiagrams commute in

$$\begin{tikzcd}
C \arrow[r] \arrow[dd, tail, "c"'] & X^{\mathbb{I}} \arrow[d, "\partial \Rightarrow X"] \\
& X \times X \arrow[d] \\
Z \arrow[r] \arrow[ur, dotted, "k"] & X
\end{tikzcd}$$

where $(\partial \Rightarrow X) : X^{\mathbb{I}} \to X \times X$ is the Leibniz exponential of X by the boundary map $\partial : 1 + 1 \to \mathbb{I}$ (and we require the condition for both projections $X \times X \to X$).

Filling and Composition

Proposition

An object X has filling if and only if it has composition.

Proof by pictures.

We want to fill the following open 2-box in X:

$$
\begin{tikzcd}
A' & B' \\
A \arrow[u, "a"] \arrow[r, "p"'] & B \arrow[u, "b"']
\end{tikzcd}
$$

$\square$

Filling and Composition

Proposition

An object X has filling if and only if it has composition.

Proof by pictures.

Make a higher dimensional composition problem using connections:

$$
\begin{tikzcd}
A' & & & B' \\
& A \arrow[ul, "a"'] \arrow[r, "p"] & B \arrow[ur, "b"] & \\
& A \arrow[u, "="] \arrow[r, "p"'] \arrow[dl, "="'] & B \arrow[u, "="'] \arrow[dr, "="] & \\
A \arrow[uuu, "a"] \arrow[rrr, "p"'] & & & B \arrow[uuu, "b"']
\end{tikzcd}
$$

□

Filling and Composition

Proposition

An object X has filling if and only if it has composition.

Proof by pictures.

Since X has composition, the (partial) open 3-box has a top face, which is a filler for the original open 2-box.

$$\begin{tikzcd}
A' & & & B' \\
& A \arrow[ul, "a"'] \arrow[r, "p"] & B \arrow[ur, "b"] & \\
& A \arrow[u, "="] \arrow[r, "p"'] \arrow[dl, "="'] & B \arrow[u, "="'] \arrow[dr, "="] & \\
A \arrow[uuu, "a"] \arrow[rrr, "p"'] & & & B \arrow[uuu, "b"']
\end{tikzcd}$$

□

Filling and Composition

Proposition

An object X has filling if and only if it has composition.

Proof.

For an algebraic proof, first use the connections to get maps in $\mathcal{E}^{\mathbb{2}}$ of the form

$$\begin{tikzcd} \delta & \delta \arrow[l, "="'] \arrow[d] \\ \delta \otimes \delta \arrow[u] & i \otimes \delta \arrow[l] \end{tikzcd}$$

where $i : 1 \to 1 + 1$.

Filling and Composition

From

$$\begin{tikzcd}
\delta & \delta \arrow[l, "="'] \arrow[d] \\
\delta \otimes \delta \arrow[u] & i \otimes \delta \arrow[l]
\end{tikzcd}$$

we then get

$$\begin{tikzcd}
\delta \Rightarrow X \arrow[r, "="] \arrow[d] & \delta \Rightarrow X \\
\delta \otimes \delta \Rightarrow X \arrow[r] \arrow[d, "\cong"'] & i \otimes \delta \Rightarrow X \arrow[u] \arrow[d, "\cong"] \\
\delta \Rightarrow (\delta \Rightarrow X) \arrow[r] & \delta \Rightarrow (i \Rightarrow X)
\end{tikzcd}$$

Filling and Composition

So for any cofibration $c : C \rightarrowtail Z$ and filling problem

$$\begin{tikzcd} C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathbb{I}} \arrow[d, "\delta \Rightarrow X"] \\ Z \arrow[r] & X \end{tikzcd}$$

we can extend by maps

$$\begin{tikzcd} C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathbb{I}} \arrow[r] \arrow[d, "\delta \Rightarrow X"] & \cdot \arrow[r] \arrow[d, "\delta \Rightarrow (\delta \Rightarrow X)"] & \cdot \arrow[r] \arrow[d, "\delta \Rightarrow (i \Rightarrow X)"] & X^{\mathbb{I}} \arrow[d, "\delta \Rightarrow X"] \\ Z \arrow[r] & X \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & X \end{tikzcd}$$

Filling and Composition

Transposing the middle section yields

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, tail, "c\otimes\delta"'] & X^{\mathbb{I}} \arrow[rr] \arrow[d, "\delta\Rightarrow X"] & & X^{1+1} \arrow[d, "i\Rightarrow X"] \\
\cdot \arrow[r] & X \arrow[rr, "="'] & & X
\end{tikzcd}$$

Filling and Composition

Transposing the middle section yields

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d, tail, "c\otimes\delta"'] & X^{\mathbb{I}} \arrow[rr] \arrow[d, "\delta\Rightarrow X"] & & X^{1+1} \arrow[d, "i\Rightarrow X"] \\
\cdot \arrow[r] \arrow[urrr, dotted] & X \arrow[rr, "="'] & & X
\end{tikzcd}
$$

which has a diagonal filler by composition, since $c \otimes \delta$ is also a cofibration.

Filling and Composition

Transposing back thus gives a diagonal filler

$$
\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathbb{I}} \arrow[r] \arrow[d, "\delta\Rightarrow X"] & \cdot \arrow[r] \arrow[d, "\delta\Rightarrow(\delta\Rightarrow X)"] & \cdot \arrow[r] \arrow[d, "\delta\Rightarrow(i\Rightarrow X)"] & X^{\mathbb{I}} \arrow[d, "\delta\Rightarrow X"] \\
Z \arrow[r] \arrow[urrr, dotted] & X \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & X
\end{tikzcd}
$$

which is a filler for the original problem

$$
\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "c"'] & X^{\mathbb{I}} \arrow[d, "\delta\Rightarrow X"] \\
Z \arrow[r] \arrow[ur, dotted] & X
\end{tikzcd}
$$

$\square$

Thus filling and composition are equivalent.

Weak equivalence

Definition

A map $f : A \longrightarrow B$ is a *weak equivalence* if it factors as a trivial cofibration followed by a trivial fibration.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, hook, "\sim"'] & & B \\ & \cdot \arrow[ur, two heads, "\sim"'] & \end{tikzcd}$$

It requires some work to show that these weak equivalences satisfy the 3 for 2 property, giving a QMS. One step of the proof uses a fibrant universe, which is our objective here.

The equivalence extension property

Definition (EEP)

The EEP says that weak equivalences extend along cofibrations $C' \rightarrowtail C$ in the following sense: given fibrations $A' \twoheadrightarrow C'$ and $B \twoheadrightarrow C$ and a weak equivalence $w' : A' \simeq C' \times_C B$ over C',

$$
\begin{tikzcd}
A' \arrow[rrr, dotted] \arrow[dr, "w'", "\sim"'] \arrow[dd, two heads] & & & A \arrow[dd, dotted, two heads] \arrow[dr, dotted, "w", "\sim"'] & \\
& C' \times_C B \arrow[rrr] \arrow[dl, two heads] & & & B \arrow[dl, two heads] \\
C' \arrow[rrr, tail] & & & C .
\end{tikzcd}
$$

there is a fibration $A \twoheadrightarrow C$ and a weak equivalence $w : A \simeq B$ over C that pulls back to w'.

This is was shown by Voevodsky for Kan simplicial sets, and it follows from our axioms.

The universe U of fibrations

Definition

A *universe of fibrations* is a small fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ such that every small fibration $A \twoheadrightarrow X$ is a pullback of $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ along a canonical classifying map $X \to \mathsf{U}$.

$$\begin{tikzcd} A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d, two heads] \\ X \arrow[r] & \mathsf{U} \end{tikzcd}$$

The universe U of fibrations

Proposition

There is a universe of fibrations.

Construction.

For any family $A \to X$ there is an *object of fibrations structures*,

$$\mathsf{Fib}(A) \longrightarrow X,$$

sections of which correspond to fibration structures on $A \to X$.
Take $\mathsf{U} \to \mathsf{V}$ to be the object of fibration structures on $\dot{\mathsf{V}} \to \mathsf{V}$,

$$\mathsf{U} = \mathsf{Fib}(\dot{\mathsf{V}}) \longrightarrow \mathsf{V}.$$

Then define $\dot{\mathsf{U}} \to \mathsf{U}$ by pulling back the universal small family.

$$\begin{tikzcd}
\dot{\mathsf{U}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{V}} \arrow[d] \\
\mathsf{U} \arrow[r] & \mathsf{V}
\end{tikzcd}$$

The universe U of fibrations

We said $\mathsf{U} = \mathsf{Fib}(\dot{\mathsf{V}})$, and we defined $\dot{\mathsf{U}} \to \mathsf{U}$ by:

$$\begin{tikzcd}
\dot{\mathsf{U}} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \dot{\mathsf{V}} \arrow[d] \\
\mathsf{U} \arrow[r] & \mathsf{V}
\end{tikzcd}$$

But $\mathsf{Fib}(-)$ is stable under pullback, so there is a section

$$\begin{tikzcd}
\dot{\mathsf{U}} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \dot{\mathsf{V}} \arrow[d] \\
\mathsf{U} \arrow[r] \arrow[d, dotted, bend right] & \mathsf{V} \\
\mathsf{Fib}(\dot{\mathsf{U}}) \arrow[r] \arrow[u] & \mathsf{Fib}(\dot{\mathsf{V}}) \arrow[u]
\end{tikzcd}$$

Thus $\dot{\mathsf{U}} \longrightarrow \mathsf{U}$ is a fibration.

The universe U of fibrations

A fibration structure α on a family $A \to X$ therefore gives rise to a factorization of the classifying map to $\dot{\mathsf{V}} \to \mathsf{V}$.

$$
\begin{tikzcd}
 & A \arrow[rr] \arrow[dd, two heads] & & \dot{\mathsf{V}} \arrow[dd] \\
\mathsf{Fib}(A) \arrow[rr] \arrow[dr] & & \mathsf{Fib}(\dot{\mathsf{V}}) \arrow[dr] & \\
 & X \arrow[rr] \arrow[ul, dotted, bend left, "\alpha"] \arrow[ur, dotted] & & \mathsf{V}
\end{tikzcd}
$$

The universe U of fibrations

A fibration structure α on a family $A \to X$ therefore gives rise to a factorization of the classifying map to $\dot{\mathsf{V}} \to \mathsf{V}$.

$$
\begin{tikzcd}
 & A \arrow[rr] \arrow[ddd] \arrow[dr, dotted] & & \dot{\mathsf{V}} \arrow[ddd] \\
 & & \dot{\mathsf{U}} \arrow[ur] \arrow[d] & \\
\mathrm{Fib}(A) \arrow[rr, crossing over] \arrow[dr] & & \mathrm{Fib}(\dot{\mathsf{V}}) \arrow[dr] & \\
 & X \arrow[rr] \arrow[ul, dotted, bend left] \arrow[ur, dotted] & & \mathsf{V}
\end{tikzcd}
$$

The universe U of fibrations

A fibration structure α on a family $A \to X$ therefore gives rise to a factorization of the classifying map to $\dot{\mathsf{V}} \to \mathsf{V}$ through the fibration classifier $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$.

$$\begin{tikzcd}
 & A \arrow[rr] \arrow[ddd] \arrow[dr, dotted] & & \dot{\mathsf{V}} \arrow[ddd] \\
 & & \dot{\mathsf{U}} \arrow[ur] \arrow[d] & \\
\mathsf{Fib}(A) \arrow[rr, crossing over] \arrow[dr] & & \mathsf{U} \arrow[dr] & \\
 & X \arrow[ul, dotted, bend left] \arrow[ur, dotted] \arrow[rr] & & \mathsf{V}
\end{tikzcd}$$

$\square$

The construction of Fib uses the *root functor* $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$.

EEP in terms of U

Given a universe of fibrations $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$, the EEP says that $\mathsf{Eq} \longrightarrow \mathsf{U}$ is a TFib:

$$
\begin{tikzcd}
A' \simeq B' \arrow[rrr] \arrow[dr] \arrow[dd, two heads] & & & \mathsf{U}_0 \simeq \mathsf{U}_1 \arrow[dl] \arrow[dd] \\
& A \simeq B \arrow[r] \arrow[dd] & \dot{\mathsf{U}} \arrow[dd] & \\
C' \arrow[rrr] \arrow[dr, hook] & & & \mathsf{Eq} \arrow[dl] \\
& C \arrow[r] \arrow[urr, dotted] & \mathsf{U} &
\end{tikzcd}
$$

This is one way of stating univalence for $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$.

U is fibrant

From univalence one can then show that the base U is fibrant.

Theorem
The universe U *is fibrant.*

Voevodsky proved this directly for Kan simplicial sets, using minimal fibrations.

Shulman gave a general proof from univalence, using 3 for 2 for weak equivalences.

Coquand gave a general proof from univalence that avoids 3 for 2 by using composition.

U is fibrant

By the reduction of filling to composition, it suffices to show:

Lemma

The universe U *has composition.*

Proof.

Consider a composition problem

$$\begin{tikzcd}
C \arrow[r] \arrow[dd, tail, "c"'] & U^{\mathbb{I}} \arrow[d] \\
 & \mathrm{U} \times \mathrm{U} \arrow[d] \\
Z \arrow[r] \arrow[ur, dotted, "k"] & \mathrm{U}
\end{tikzcd}$$

□

U is fibrant

The canonical map $\mathsf{U}^{\mathbb{I}} \longrightarrow \mathsf{U} \times \mathsf{U}$ factors (over $\mathsf{U} \times \mathsf{U}$) through the object Eq of equivalences via $i = \mathsf{IdtoEq}$,[†]

$$
\begin{tikzcd}
C \arrow[r] \arrow[ddd, tail, "c"'] & U^{\mathbb{I}} \arrow[d, "i"] \\
 & \mathsf{Eq} \arrow[d] \\
 & \mathsf{U} \times \mathsf{U} \arrow[d] \\
Z \arrow[r] \arrow[ur, dotted, "k"] & \mathsf{U}
\end{tikzcd}
$$

[†] there is more to be said here, time permitting

U is fibrant

The canonical map $\mathrm{U}^{\mathbb{I}} \longrightarrow \mathrm{U} \times \mathrm{U}$ factors (over $\mathrm{U} \times \mathrm{U}$) through the object Eq of equivalences via $i := \mathrm{IdtoEq}$,[†]

$$\begin{tikzcd}
C \arrow[r] \arrow[ddd, tail, "c"'] & U^{\mathbb{I}} \arrow[d, "i"] \\
 & \mathrm{Eq} \arrow[d] \\
 & \mathrm{U} \times \mathrm{U} \arrow[d] \\
Z \arrow[r] \arrow[uur, dotted, "j"] \arrow[ur, dotted, "k"] & \mathrm{U}
\end{tikzcd}$$

But the projection $\mathrm{Eq} \longrightarrow \mathrm{U}$ is a trivial fibration by univalence, so there is a diagonal filler j. Composing gives the required k. $\square$

[†]there is more to be said here, time permitting

Done!