Categories and Bishop sets

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ABM 2019
Munich, 03.05.2019

Bishop 1967

A set is defined by describing what must be done to construct an element of the set, and what must be done to show that two elements of the set are equal.

An older definition

A set of elements belonging to some conceptual sphere is called well-defined if, on the basis of its definition . . . it must be regarded as internally determined, both whether any object of that conceptual sphere belongs as an element to the mentioned set, and also whether two objects belonging to the set, in spite of formal differences in the mode of givenness, are equal to each other or not.

Cantor 1882

A set of elements belonging to some conceptual sphere is called well-defined if, on the basis of its definition and in accordance with the logical principle of the excluded third, it must be regarded as internally determined, both whether any object of that conceptual sphere belongs as an element to the mentioned set, and also whether two objects belonging to the set, in spite of formal differences in the mode of givenness, are equal to each other or not.

A natural question:

Is Bishop's informal description of a set enough to provide a formal account of it within a suitable formalization of BISH?

If not, what kind of choices are the most compatible to BISH?

Feferman 1979

Let T be a formal theory of an informal body of mathematics M.

(i) T is adequate for M, if every concept, argument, and result of M is represented by a (basic or defined) concept, proof, and a theorem, respectively, of T.

(ii) T is faithful to M, if every basic concept of T corresponds to a basic concept of M and every axiom and rule of T corresponds to or is implicit in the assumptions and reasoning followed in M (i.e., T does not go beyond M conceptually or in principle)

(iii) (Beeson 1981) T is suitable to M, if T is adequate for M and faithful to M.

The standard interpretation of Bishop sets is within MLTT

The standard way to understand a Bishop set A is through a setoid in MLTT i.e., a type A in a fixed universe $\mathcal{U}$ equipped with a term $\simeq: A \rightarrow A \rightarrow \mathcal{U}$ (eqrel).

Even if we translate a Bishop set as a set in CZF, we get back to setoids through Aczel's interpretation of CZF into MLTT.

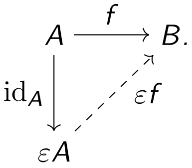
Is the theory of setoids a suitable formalization of Bishop's set theory?

It doesn't seem faithful: the J-rule not in BISH

If $A : \mathcal{U}$, then $=_A$ is the least reflexive relation on A (J-rule) and the free setoid on A is $\varepsilon A := (A, =_A)$.

Proposition (Universal property of free setoid)

For every $(B, \sim_B)$ and every function $f : A \to B$, there is a setoid-map $\varepsilon f : \varepsilon A := A \to B$ sttfdc

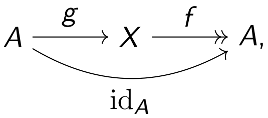


Proof.

Let $(\varepsilon f)(a) := f(a)$, and since $=_B$ is the least reflexive rel on B,

$$a =_A a' \to (\varepsilon f)(a) =_B (\varepsilon f)(a') \to f(a) \sim_B f(a').$$

A is a choice set(oid) iff every $f : X \twoheadrightarrow A$, has a right inverse g of f



"Every set is a choice set" $\Leftrightarrow \mathrm{AC}$.

Proposition

$(A, =_A)$ is a choice set.

Proof.

$$\prod_{a:A} \sum_{x:X} (f(x) =_A a) \to \sum_{g:A \to X} \prod_{a:A} f(g(a)) =_A a$$

$$a =_A a' \to g(a) =_X g(a') \to g(a) \sim_X a'.$$

□

Corollary

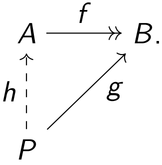
Every setoid is a quotient of a choice set.

Proof.

$q : (A, =_A) \twoheadrightarrow (A, \sim_A)$, $a \mapsto a$, and $a =_A a' \to a \sim_A a'$. □

The presentation axiom

If $\boldsymbol{C}$ is a category and P in C_0, then P is projective, if $\forall_{A,B \in C_0} \forall_{f : A \twoheadrightarrow B} \forall_{g : P \to B} \exists_{h : P \to A}$ sttfdc



Presentation axiom (PAx) in $\boldsymbol{C}$: $\boldsymbol{C}$ *has enough projectives* i.e., for every object C in $\boldsymbol{C}$ there is $f : P \twoheadrightarrow C$, where P is projective.

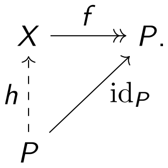
$\mathrm{PAx} \Rightarrow \mathrm{DC}$, ($\mathcal{M} \models \mathrm{ZF} + \mathrm{DC} + \neg\mathrm{AC}$ and $\mathcal{M} \not\models \mathrm{PAx}$).

Not in Aczel's CZF.
$\mathrm{ZF} \vdash (\mathrm{PAx} \Rightarrow \mathrm{AC})$?
$\mathrm{IZF} + \mathrm{PAx} \not\vdash \mathrm{AC}$.
$\mathrm{CZF} + \mathrm{AC} \vdash \mathrm{REM}$, $\mathrm{IZF} + \mathrm{AC} \vdash \mathrm{PEM}$, $\mathrm{IZF} + \mathrm{PAx}$ implies no form of PEM.
(See Rathjen 2006).

Proposition

A projective setoid $(P, \sim_P)$ *is a choice set.*

Proof.



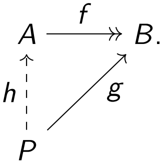
$\square$

Proposition (Palmgren)

A choice set(oid) $(P, \sim_P)$ is a projective setoid.

Proof.

Let f, g, we want to define h sttfdc



$$Q := \sum_{(a,p):A\times P} f(a) =_B g(p)$$

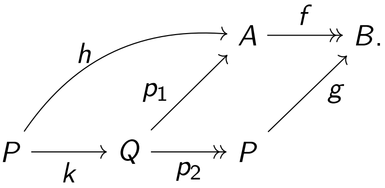
$$p_1 : Q \to A, \quad p_1(a, p, e) := a,$$

$$p_2 : Q \to P, \quad p_2(a, p, e) := p,$$

$$f \circ p_1 = g \circ p_2.$$

proof continued

Since $p_2 : Q \twoheadrightarrow P$ and P is a choice set, there is $k : P \to Q$ s.t. $p_2 \circ k = \mathrm{id}_P$. If $h := p_1 \circ k$,



$$f \circ (p_1 \circ k) = (f \circ p_1) \circ k = (g \circ p_2) \circ k = g \circ (p_2 \circ k) = g \circ \mathrm{id}_P = g.$$

Corollary

PAx *holds for setoids.*

Proof.

Every setoid is the surjective image of a choice set, hence of a projective setoid. □

It seems that

Setoids in a universe $\mathcal{U}$ do not form a faithful formalization of Bishop sets. They have properties that Bishop sets are not expected to have.

Moreover

(Moerdijk-Palmgren, 2000) Setoids with a hierarchy of universes is the standard model of a ΠW-pretopos (:$\Leftrightarrow$ lccc pretopos with W-types $\Leftrightarrow$ exact ML-category).

A ΠW-pretopos is closed under exact completion (BvdBerg),

Most toposes are not,

Hence, there are many ΠW-pretoposes that are not toposes!

$$\mathrm{MLTT} + W\text{-types} \sim \mathrm{CZF} + \mathrm{REA}$$

$$\mathrm{BISH}^* := \mathrm{BISH} + \text{inductive definitions with rules of countably many premisses}$$

is much weaker.

Bishop's theory of sets (Chapter 3 of Bi67 and BB85) has left many issues unsettled ($\mathcal{P}(X), \texttt{Fam}(I)$, dependency).

Richman's mixture of category theory and Bishop's set theory in MRR88 is underdeveloped too: no explanation is provided for understanding categories within BISH.

BST is a reconstruction of Bishop's theory of sets, that will help us formulate an adequate and faithful formalization of the latter, and hopefully answer the following question too:

What kind of category is abstracted from Bishop sets?

(categorical characterization of Bishop sets $\approx$ Bishop setoids)
Palmgren 2012: CETCS, a predicative and constructive variation of Lawvere's ETCS.

Primitives of BST I

1. (s, t).
2. equality $:=$ between terms.
3. $\mathtt{pr}_1(s, t) := s$ and $\mathtt{pr}_2(s, t) := t$.
4. $\mathbb{N}$.
5. Any other totality X is defined through a "membership-formula" $x \in X$.
6. A defined equality on X is a formula $x =_X y$ that satisfies the properties of an equivalence relation.
7. If X is a set and Y is a totality, an assignment routine $\alpha : X \rightsquigarrow Y$ from X to Y is a finite routine assigning an element y of Y, to each given element x of X. In this case we write $\alpha(x) := y$.
8. If X, Y are sets, an assignment routine $f : X \rightsquigarrow Y$ is a function, if $f(x) =_Y f(x')$, for every $x, x' \in X$, such that $x =_X x'$. In this case we write $f : X \to Y$.

Primitives of BST II

1. $\mathbb{F}(X, Y)$ with pointwise equality is a set (function extensionality).
2. The (univalent) universe of sets $\mathbb{V}_0$ with equality

   $$X =_{\mathbb{V}_0} Y :\Leftrightarrow \exists_{f \in \mathbb{F}(X,Y)} \exists_{g \in \mathbb{F}(Y,X)} \big(g \circ f = \mathrm{id}_X \ \& \ f \circ g = \mathrm{id}_Y \big)$$

   is a class.
3. If I is a set and $\mu_0 : I \rightsquigarrow \mathbb{V}_0$, a dependent assignment routine over μ_0 is an assignment routine μ_1 that assigns to each element i in I an element $\mu_1(i)$ in $\mu_0(i)$. We denote such a routine by

   $$\mu_1 : \bigcurlywedge_{i \in I} \mu_0(i),$$

   and their totality by $\mathbb{A}(I, \mu_0)$. If $\mu_1, \nu_1 : \bigcurlywedge_{i \in I} \mu_0(i)$, we define

   $$\mu_1 =_{\mathbb{A}(I,\mu_0)} \nu_1 :\Leftrightarrow \forall_{i \in I} \big(\mu_1(i) =_{\mu_0(i)} \nu_1(i) \big).$$

If I is a set, and $D(I) := \{(i,j) \in I \times I \mid i =_I j\}$, a family of sets indexed by I is a pair $\Lambda := (\lambda_0, \lambda_1)$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, and

$$\lambda_1 : \bigcurlywedge_{(i,j) \in D(I)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big),$$

such that, if $\lambda_1(i,j) := \lambda_{ij}$, for every $(i,j) \in D(I)$,

(a) For every $i \in I$, we have that $\lambda_{ii} := \mathrm{id}_{\lambda_0(i)}$.

(b) If $i =_I j$ and $j =_I k$, the following diagram commutes

$$\begin{array}{ccc} \lambda_0(i) & & \\ \downarrow \lambda_{ij} & \searrow \lambda_{ik} & \\ \lambda_0(j) & \xrightarrow[\lambda_{jk}]{} & \lambda_0(k). \end{array}$$

If $i =_I j$, we call the function λ_{ij} the transport map from $\lambda_0(i)$ to $\lambda_0(j)$. and we call λ_1 the modulus of function-likeness of λ_0:

$$(\lambda_{ij}, \lambda_{ji}) : \lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j).$$

Let $\Lambda := (\lambda_0, \lambda_1)$ and $M := (\mu_0, \mu_1)$ be I-families of sets. A family-map from Λ to M is a d.a.r.

$$\Psi : \bigcurlywedge_{i \in I} \mathbb{F}\big(\lambda_0(i), \mu_0(i)\big)$$

such that for every $(i, j) \in D(I)$ tfdc

$$\begin{array}{ccc} \lambda_0(i) & \xrightarrow{\lambda_{ij}} & \lambda_0(j) \\ {\scriptstyle \Psi_i}\downarrow & & \downarrow{\scriptstyle \Psi_j} \\ \mu_0(i) & \xrightarrow[\mu_{ij}]{} & \mu_0(j). \end{array}$$

$\mathtt{Map}_I(\Lambda, M)$ with $\Psi =_{\mathtt{Map}_I(\Lambda,M)} \Xi :\Leftrightarrow \forall_{i \in I}\big(\Psi_i =_{\mathbb{F}(\lambda_0(i),\mu_0(i))} \Xi_i\big)$.

$\mathtt{Fam}(I)$ the totality of I-families of sets with equality

$$\Lambda =_{\mathtt{Fam}(I)} M :\Leftrightarrow \exists_{\Phi \in \mathtt{Map}_I(\Lambda,M)} \exists_{\Xi \in \mathtt{Map}_I(M,\Lambda)} \big(\Phi \circ \Xi = \mathrm{id}_M \;\&\; \Xi \circ \Phi = \mathrm{id}_\Lambda\big).$$

Let $\Lambda := (\lambda_0, \lambda_1)$ be an I-family of sets.

The exterior union $\sum_{i \in I} \lambda_0(i)$ of Λ is defined by

$$w \in \sum_{i \in I} \lambda_0(i) :\Leftrightarrow \exists_{i \in I} \exists_{x \in \lambda_0(i)} \big(w := (i, x)\big),$$

$$(i, x) =_{\sum_{i \in I} \lambda_0(i)} (j, y) :\Leftrightarrow i =_I j \ \& \ \lambda_{ij}(x) =_{\lambda_0(j)} y.$$

The totality $\prod_{i \in I} \lambda_0(i)$ of dependent functions over Λ is defined by

$$\Phi \in \prod_{i \in I} \lambda_0(i) :\Leftrightarrow \Phi \in \mathbb{A}(I, \lambda_0) \ \& \ \forall_{(i,j) \in D(I)} \big(\Phi_j =_{\lambda_0(j)} \lambda_{ij}(\Phi_i)\big),$$

and it is equipped with the equality of $\mathbb{A}(I, \lambda_0)$.

Proposition

Let $\Lambda := (\lambda_0, \lambda_1)$, $M := (\mu_0, \mu_1) \in \texttt{Fam}(I)$, *and* $\Psi \in \texttt{Map}_I(\Lambda, M)$.

(i) *For every* $i \in I$ *the a.r.* $e_i^{\Lambda} : \lambda_0(i) \rightsquigarrow \sum_{i\in I} \lambda_0(i)$, *defined by* $x \mapsto (i, x)$, *is an embedding of* $\lambda_0(i)$ *into* $\sum_{i\in I} \lambda_0(i)$.

(ii) *The a.r.* $\Sigma\Psi : \sum_{i\in I} \lambda_0(i) \rightsquigarrow \sum_{i\in I} \mu_0(i)$,

$$\Sigma\Psi(i, x) := (i, \Psi_i(x)),$$

is a function from $\sum_{i\in I} \lambda_0(i)$ *to* $\sum_{i\in I} \mu_0(i)$, *s.t. for every* $i \in I$ *tfdc*

$$\begin{array}{ccc} \lambda_0(i) & \xrightarrow{\ \Psi_i\ } & \mu_0(i) \\ {\scriptstyle e_i^{\Lambda}} \downarrow & & \downarrow {\scriptstyle e_i^{M}} \\ \sum_{i\in I} \lambda_0(i) & \xrightarrow[\Sigma\Psi]{} & \sum_{i\in I} \mu_0(i). \end{array}$$

(iii) *If every* Ψ_i *is an embedding, then* $\Sigma\psi$ *is an embedding.*

Proposition

Let $\Lambda := (\lambda_0, \lambda_1)$, $M := (\mu_0, \mu_1) \in \mathtt{Fam}(I)$, *and* $\Psi \in \mathtt{Map}_I(\Lambda, M)$.

(i) For every $i \in I$ *the a.r.* $\pi_i^{\Lambda} : \prod_{i \in I} \lambda_0(i) \rightsquigarrow \lambda_0(i)$, *defined by* $\Theta \mapsto \Theta_i$, *is a function from* $\prod_{i \in I} \lambda_0(i)$ *to* $\lambda_0(i)$.

(ii) The a.r. $\Pi\Psi : \prod_{i \in I} \lambda_0(i) \rightsquigarrow \prod_{i \in I} \mu_0(i)$,

$$[\Pi\Psi(\Theta)]_i := \Psi_i(\Theta_i),$$

is a function from $\prod_{i \in I} \lambda_0(i)$ *to* $\prod_{i \in I} \mu_0(i)$, *s.t. for every* $i \in I$ *tfdc*

$$\begin{array}{ccc} \lambda_0(i) & \xrightarrow{\Psi_i} & \mu_0(i) \\ \uparrow{\scriptstyle \pi_i^{\Lambda}} & & \uparrow{\scriptstyle \pi_i^{M}} \\ \prod_{i \in I} \lambda_0(i) & \xrightarrow[\Pi\Psi]{} & \prod_{i \in I} \mu_0(i). \end{array}$$

(iii) If every Ψ_i *is an embedding, then* $\Pi\Psi$ *is an embedding.*

Distributivity of $\prod$ over $\sum$ (In Iccc, Martin-Löf, Awodey)

Let X, Y be sets,

(ρ_0, ρ_1) is an $(X \times Y)$-family of sets,

If $x \in X$, let $(\lambda_0^x, \lambda_1^x)$ is the Y-family

$$\lambda_0^x(y) := \rho_0(x, y), \ \ \lambda_1^x : \rho_0(x, y) \to \rho_0(x, y'), \ \ \lambda_{yy'}^x := \rho_{(x,y)(x,y')}$$

Let the X-family of sets (μ_0, μ_1), where

$$\mu_0(x) := \sum_{y \in Y} \rho_0(x, y),$$

$$\mu_1 : \bigcurlywedge_{(x,x') \in D(X)} \mathbb{F}\left(\sum_{y \in Y} \rho_0(x, y), \sum_{y \in Y} \rho_0(x', y)\right)$$

$$\mu_{xx'}(y, u) := \big(y, \rho_{(x,y)(x',y)}(u)\big).$$

Lemma

If $\Phi \in \prod_{x\in X} \mu_0(x) := \prod_{x\in X} \sum_{y\in Y} \rho_0(x,y)$*, the a.r.* $f_\Phi : X \rightsquigarrow Y$*,* $x \mapsto \mathtt{pr}_1(\Phi_x)$*, is a function from* X *to* Y*.*

Lemma

If $f : X \to Y$ *the pair* $\left(\nu_0^f, \nu_1^f\right)$ *is an* X*-family of sets, where*

$$\nu_0(x) := \rho_0(x, f(x)),$$

$$\nu_1^f : \bigcurlywedge_{(x,x')\in D(X)} \mathbb{F}\left(\rho_0(x, f(x)), \rho_0(x', f(x'))\right), \quad \nu_{xx'}^f := \rho_{(x,f(x))(x',f(x'))}.$$

Lemma

The pair (ξ_0, ξ_1) *is an* $\mathbb{F}(X, Y)$*-family of sets, where*

$$\xi_0(f) := \prod_{x\in X} \nu_0^f(x) := \prod_{x\in X} \rho_0(x, f(x)),$$

$$\xi_{ff'} : \prod_{x\in X} \rho_0(x, f(x)) \to \prod_{x\in X} \rho_0(x, f'(x)), [\xi_{ff'}(H)]_x := \rho_{(x,f(x))(x,f'(x))}(H_x)$$

Theorem

If

$$\Phi \in \prod_{x \in X} \mu_0(x) := \prod_{x \in X} \sum_{y \in Y} \rho_0(x, y),$$

there is

$$\Theta_\Phi \in \prod_{x \in X} \nu_0^{f_\Phi} := \prod_{x \in X} \rho_0(x, f_\Phi(x))$$

such that

$$\left(f_\Phi, \Theta_\Phi\right) \in \sum_{f \in \mathbb{F}(X,Y)} \prod_{x \in X} \rho_0(x, f_\Phi(x))$$

and the following a.r. is a function:

$$\mathrm{ac} : \prod_{x \in X} \sum_{y \in Y} \rho_0(x, y) \rightsquigarrow \sum_{f \in \mathbb{F}(X,Y)} \prod_{x \in X} \rho_0(x, f(x)),$$

$$\Phi \mapsto \left(f_\Phi, \Theta_\Phi\right).$$

Richman's categories in BISH

Definition

If I is a set, the equality-category of I has objects the elements of I, and if $i, j, k \in I$, then

$$\mathtt{Hom}_{=_I}(i, j) := \{x \in \{0\} \mid i =_I j\},$$

$$1_i := 0,$$

$$\frac{y \in \mathtt{Hom}_{=_I}(j, k) \quad \& \quad x \in \mathtt{Hom}_{=_I}(i, j)}{y \circ x := 0 \in \mathtt{Hom}_{=_I}(i, k)}.$$

An equality-functor from a set I *to a set* J is a pair $\Phi := (\phi_0, \phi_1)$, where $\phi_0 : I \rightsquigarrow J$ and

$$\phi_1 : \bigcurlywedge_{i, i' \in I} \mathbb{F}\big(\mathtt{Hom}_{=_I}(i, i'), \mathtt{Hom}_{=_J}(\phi_0(i), \phi_0(i'))\big).$$

Remark

Let I, J be sets.

(i) *If $\Phi := (\phi_0, \phi_1)$ is an equality-functor from I to J, ϕ_0 is a function from I to J.*

(ii) *If $f \in \mathbb{F}(I, J)$, there is an equality-functor $\Phi^f := \left(\phi_0^f, \phi_1^f\right)$ such that $\phi_0^f := f$.*

The equality-category of the universe of sets $\mathbb{V}_0$ has objects its elements and if $X, Y \in \mathbb{V}_0$, we define

$$\mathtt{Hom}_{=_{\mathbb{V}_0}}(X, Y) := \big\{(f, f') : \mathbb{F}(X, Y) \times \mathbb{F}(Y, X) \mid (f, f') : X =_{\mathbb{V}_0} Y\big\},$$

$$1_X := (\mathrm{id}_X, \mathrm{id}_X),$$

$$\frac{(f, f') \in \mathtt{Hom}_{=_{\mathbb{V}_0}}(X, Y) \ \ \& \ \ (g, g') \in \mathtt{Hom}_{=_{\mathbb{V}_0}}(Y, Z)}{(g \circ f, f' \circ g') \in \mathtt{Hom}_{=_{\mathbb{V}_0}}(X, Z)}.$$

An equality-functor from a set I to $\mathbb{V}_0$ is a pair $\Phi := (\phi_0, \phi_1)$, where $\phi_0 : I \rightsquigarrow \mathbb{V}_0$ and

$$\phi_1 : \bigcurlywedge_{i,j \in I} \mathbb{F}\Big(\mathtt{Hom}_{=_I}(i, j), \mathtt{Hom}_{=_{\mathbb{V}_0}}(\phi_0(i), \phi_0(j))\Big)$$

such that the following hold:

(a) For every $i \in I$, $[\phi_1(i, i)](1_i) := 1_{\phi_0(i)}$.

(b) For every $i, j, k \in I$, if $x \in \mathtt{Hom}_{=_I}(i, j)$ and $y \in \mathtt{Hom}_{=_I}(j, k)$, then

$$[\phi_1(i, k)](y \circ x) = [\phi_1(j, k)](y) \circ [\phi_1(i, j)](x).$$

Remark

Let I be a set.

(*i*) *If* $\Phi := (\phi_0, \phi_1)$ *is an equality-functor from I to* $\mathbb{V}_0$, *then* $\Lambda^{\Phi} := (\lambda_0^{\Phi}, \lambda_1^{\Phi})$, *where*

$$\lambda_0^{\Phi}(i) := \phi_0(i) \quad \& \quad \lambda_1^{\Phi}(i,j) := \mathtt{pr}_{\mathbb{F}(\phi_0(i), \phi_0(j))}\big([\phi_1(i,j)](0)\big),$$

for every $i \in I$ *and every* $(i,j) \in D(I)$, *is an I-family of sets.*

(*ii*) *If* $\Lambda := (\lambda_0, \lambda_1)$ *is an I-family of sets, then* $\Phi^{\Lambda} := (\phi_0^{\Lambda}, \phi_1^{\Lambda})$, *where*

$$\phi_0^{\Lambda}(i) := \lambda_0(i) \quad \& \quad [\phi_1^{\Lambda}(i,j)](x) := (\lambda_{ij}, \lambda_{ji}),$$

for every $i, j \in I$ *and every* $x \in \mathtt{Hom}_I(i,j)$, *is an equality-functor from I to* $\mathbb{V}_0$.

(iii) A family-map from Λ to M is a natural transformation from the functor Φ^{Λ} to the functor Φ^{M}.

Remark

Let I be a set and $i_0 \in I$. If $\mathcal{Y}^{i_0} := (y_0^{i_0}, y_1^{i_0})$, where $y_0^{i_0} : I \rightsquigarrow \mathbb{V}_0$ is defined by $y_0^{i_0}(i) := \mathtt{Hom}_{=_I}(i, i_0)$, for every $i \in I$, and

$$y_1^{i_0} : \bigwedge_{(i,j)\in D(I)} \mathbb{F}\big(\mathtt{Hom}_{=_I}(i, i_0), \mathtt{Hom}_{=_I}(j, i_0)\big)$$

$$y_1^{i_0}(i,j) := y_{ij} : \mathtt{Hom}_{=_I}(i, i_0) \to \mathtt{Hom}_{=_I}(j, i_0)$$

$$y_{ij}(x) := x,$$

for every $(i,j) \in D(I)$ and $x \in \mathtt{Hom}_{=_I}(i, i_0)$, is an I-family of sets. Moreover, if $i =_I j$, the d.a.r.

$$K_{ij} : \bigwedge_{k\in I} \mathbb{F}\big(\mathcal{Y}_0^i(k), \mathcal{Y}_0^j(k)\big),$$

$$K_{ij}(k) : \mathtt{Hom}_{=_I}(k, i) \to \mathtt{Hom}_{=_I}(k, j)$$

$$x \mapsto x,$$

is in $\mathtt{Map}_I(\mathcal{Y}^i, \mathcal{Y}^j)$.

Yoneda lemma for $\mathtt{Fam}(I)$

Theorem

If $I \in \mathbb{V}_0$, $i_0 \in I$, and $\Lambda \in \mathtt{Fam}(I)$, the are maps

$$e_{i_0,\Lambda} : \mathtt{Map}_I(\mathcal{Y}^{i_0}, \Lambda) \to \lambda_0(i_0)$$

$$\varepsilon_{i_0,\Lambda} : \lambda_0(i_0) \to \mathtt{Map}_I(\mathcal{Y}^{i_0}, \Lambda),$$

$$(e_{i_0,\Lambda}, \varepsilon_{i_0,\Lambda}) : \mathtt{Map}_I(\mathcal{Y}^{i_0}, \Lambda) =_{\mathbb{V}_0} \lambda_0(i_0).$$

If $i =_I j$, $M \in \mathtt{Fam}(I)$, and $\Psi \in \mathtt{Map}(\Lambda, M)$, tfdc

$$\begin{array}{ccc} \mathtt{Map}_I(Y^i, \Lambda) & \xrightarrow{e_{i,\Lambda}} & \lambda_0(i) \\ {\scriptstyle \mathtt{map}_I(K_{ji}, \Psi)} \downarrow & & \downarrow {\scriptstyle \Psi_j \circ \lambda_{ij}} \\ \mathtt{Map}_I(Y^j, M) & \xrightarrow[e_{j,M}]{} & \mu_0(j). \end{array}$$

Hence, $\mathtt{Map}_I(\mathcal{Y}^i, \mathcal{Y}^j) =_{\mathbb{V}_0} \mathtt{Hom}_{=_I}(i, j)$.

Let $\Lambda := (\lambda_0, \lambda_1)$ and $M := (\mu_0, \mu_1)$ be I-families of sets.

A family of Bishop topologies associated to Λ is a pair $\Phi^\Lambda := \left(\phi_0^\Lambda, \phi_1^\Lambda\right)$, where $\phi_0^\Lambda : I \rightsquigarrow \mathbb{V}_0$, and

$$\phi_1^\Lambda : \bigcurlywedge_{(i,j) \in D(I)} \mathbb{F}(\phi_0^\Lambda(i), \phi_0^\Lambda(j)),$$

(i) $\phi_0^\Lambda(i) := F_i$ and $\mathcal{F}_i := (\lambda_0(i), F_i)$ is a Bishop space.

(ii) $\lambda_{ij} \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{F}_j)$, for every $(i, j) \in D(I)$.

(iii) $\phi_1^\Lambda(i, j) := \lambda_{ji}^* : F_i \to F_j$, for every $(i, j) \in D(I)$.

$S := \left(\lambda_0, \lambda_1, \phi_0^\Lambda, \phi_1^\Lambda\right)$ is called a spectrum over I with Bishop spaces $\mathcal{F}_i$ and Bishop isomorphisms λ_{ij}.

If $T := \left(\mu_0, \mu_1, \phi_0^M, \phi_1^M\right)$ is an I-spectrum with Bishop spaces $\mathcal{G}_i$ and Bishop isomorphisms μ_{ij}, a spectrum-map from S to T is a family-map Ψ from Λ to M.

A spectrum-map Ψ from S to T is continuous, if $\Psi_i : \lambda_0(i) \to \mu_0(i)$ is in $\mathrm{Mor}(\mathcal{F}_i, \mathcal{G}_i)$, for every $i \in I$.

Remark

Let $S := (\lambda_0, \lambda_1, \phi_0^{\Lambda}, \phi_1^{\Lambda})$ *be an* I*-spectrum with Bishop spaces* $\mathcal{F}_i$ *and Bishop isomorphisms* λ_{ij}. *If* $\Theta \in \prod_{i \in I} F_i$, *the a.r.*

$$f_\Theta : \Big(\sum_{i \in I} \lambda_0(i)\Big) \rightsquigarrow \mathbb{R},$$

$$f_\Theta(i, x) := \Theta_i(x),$$

is a function from $\sum_{i \in I} \lambda_0(i)$ *to* $\mathbb{R}$.

Proof.

If

$$(i, x) =_{\sum_{i \in I} \lambda_0(i)} (j, y) :\Leftrightarrow i =_I j \;\&\; \lambda_{ij}(x) =_{\lambda_0(j)} y,$$

then by the definition of a dependent function over a set-indexed family of sets

$$f_\Theta(i, x) := \Theta_i(x) =_{\mathbb{R}} [\lambda_{ij}^*(\Theta_j)](x) := [\Theta_j \circ \lambda_{ij}](x) =_{\mathbb{R}} \Theta_j(y) := f_\Theta(j, y).$$

□

Let $S := (\lambda_0, \lambda_1, \phi_0^{\Lambda}, \phi_1^{\Lambda})$ be an I-spectrum with Bishop spaces $\mathcal{F}_i$ and Bishop isomorphisms λ_{ij}. The sum Bishop space of S is the pair

$$\sum_{i \in I} \mathcal{F}_i := \Big(\sum_{i \in I} \lambda_0(i), \sum_{i \in I} F_i \Big), \quad \text{where} \quad \sum_{i \in I} F_i := \bigvee_{\Theta \in \prod_{i \in I} \varphi_0^{\Lambda}(i)} f_{\Theta},$$

and the dependent product Bishop space of S is the pair

$$\prod_{i \in I} \mathcal{F}_i := \Big(\prod_{i \in I} \lambda_0(i), \prod_{i \in I} F_i \Big), \quad \text{where} \quad \prod_{i \in I} F_i := \bigvee_{i \in I}^{f \in F_i} (f \circ \pi_i^{\Lambda}),$$

and π_i^{Λ} is the projection function.

Proposition

Let $S := (\lambda_0, \lambda_1, \phi_0^{\Lambda}, \phi_1^{\Lambda})$ *be an* I *spectrum with Bishop spaces* $\mathcal{F}_i$ *and Bishop isomorphisms* λ_{ij}*,* $T := (\mu_0, \mu_1, \phi_0^{M}, \phi_1^{M})$ *an* I*-spectrum with Bishop spaces* $\mathcal{G}_i$ *and Bishop isomorphisms* μ_{ij}*, and* Ψ *a spectrum-map from* S *to* T*.*

(i) *If* $i \in I$*, then* $e_i^{\Lambda} \in \mathrm{Mor}\big(\mathcal{F}_i, \sum_{i \in I} \mathcal{F}_i\big)$*.*

(ii) *If* Ψ *is continuous, then* $\Sigma\Psi \in \mathrm{Mor}\big(\sum_{i \in I} \mathcal{F}_i, \sum_{i \in I} \mathcal{G}_i\big)$*.*

(iii) *If* Ψ *is continuous, then* $\Pi\Psi \in \mathrm{Mor}\big(\prod_{i \in I} \mathcal{F}_i, \prod_{i \in I} \mathcal{G}_i\big)$*.*

Directed families of sets

Let $(I, \prec_I)$ be a directed set. A directed family of sets indexed by $(I, \prec_I)$ is a pair $\Lambda^{\prec} := (\lambda_0, \lambda_1^{\prec})$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, and

$$\lambda_1^{\prec} : \bigcurlywedge_{(i,j) \in \prec(I)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big),$$

(a) For every $i \in I$, we have that $\lambda_{ii}^{\prec} := \mathrm{id}_{\lambda_0(i)}$.

(b) If $i \prec_I j$ and $j \prec_I k$, the following diagram commutes

$$\begin{array}{ccc} \lambda_0(i) & & \\ \downarrow{\scriptstyle \lambda_{ij}^{\prec}} & \overset{\lambda_{ik}^{\prec}}{\searrow} & \\ \lambda_0(j) & \xrightarrow[\lambda_{jk}^{\prec}]{} & \lambda_0(k). \end{array}$$

$$(i, x) =_{\sum_{i \in I}^{\prec} \mu_0(i)} (j, y) :\Leftrightarrow \exists_{k \in I}\big(i \prec k \;\&\; j \prec k \;\&\; \lambda_{ik}^{\prec}(x) =_{\lambda_0(k)} \lambda_{jk}^{\prec}(y)\big).$$

Is the theory of setoids an adequate formalization of Bishop's set theory?

In MLTT $\texttt{Fam}(I)$ corresponds to the type $I \to \mathcal{U}$, which "belongs" to the successor universe $\mathcal{U}'$ of $\mathcal{U}$.

If $\texttt{Fam}(I)$ was a Bishop set, the constant I-family with value $\texttt{Fam}(I)$ would be defined though a totality in which it belongs to. From a predicative point of view, this cannot be accepted.

Moreover, the equality of $\texttt{Fam}(I)$, as that of $\mathbb{V}_0$, has computational content, and the category that naturally corresponds to $\texttt{Fam}(I)$ has non-trivial $\texttt{Hom}(\Lambda, M)$.

On the other hand, $\texttt{Fam}(I)$ is not a class, like $\mathbb{V}_0$, or $\mathcal{P}(X)$.

$\texttt{Fam}(I)$ as an impredicative set?

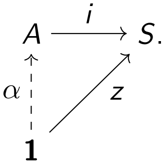
Bishop's informal definition cannot distinguish between sets, classes, and impredicative sets.

Hence, Bishop's description of the notion of set is incomplete for the needs of (formalisation of) BISH.

CETCS, Palmgren 2012, constructive version of Lawvere's ETCS

1. $\boldsymbol{C}$ is cartesian: it has a terminal object $\mathbf{1}$, it has products and equalizers.
2. $\boldsymbol{C}$ is cocartesian: it has an initial object $\mathbf{0}$, it has sums and coequalizers.
3. (Π) $\boldsymbol{C}$ has dependent products.
4. (G) An onto and monic arrow is an iso.
5. (PAx) For every object A there is an onto arrow $P \to A$, where P is a choice object.
6. $\mathbf{0}$ has no elements.

1. (DP) If $A \xrightarrow{i} S \xleftarrow{j} B$ is a sum diagram, then for every $z \in S$ we have that $z \varepsilon i$ or $z \varepsilon j$.



2. (NT) If $\mathbf{1} \xrightarrow{x} S \xleftarrow{y} \mathbf{1}$ is a sum diagram, then $x \neq y$.
3. (Fct) Any f is factored as ie, where i is a mono and e is onto.
4. (Eff) All equivalence relations are effective.

Replace PAx with DC, and G with the weaker G_0 and G_1:

(G_0) For any $f, g : A \to B$, if $\forall_{x \in A}(fx = gx)$, then $f = g$.

(G_1) An arrow $f : A \to B$ is monic iff $\forall_{x,y \in A}(fx = fy \Rightarrow x = y)$.

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