Category Theory in Explicit Mathematics

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ABM

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Overview

- Explicit Mathematics
- Category Theory
- Towards a *Category of Sets* in Explicit Mathematics. Three categories and some selected properties.
  - **EC**
    - Extensiveness
  - **ECB**: implicit Bishop Sets
    - Regularity
    - Exactness
  - $\mathbf{EC}_{\mathrm{ex}}$: explicit Bishop Sets
- Universes in Explicit Mathematics and in category theory

Explicit Mathematics

- Hilbert-style System with two kinds of variables first developed by Feferman.

- Formulas built from

$$\begin{aligned}
&\varphi \wedge \psi, \quad \varphi \vee \psi, \quad \varphi \Rightarrow \psi, \\
&\varphi = \psi, \quad \neg\varphi, \\
&\exists x \varphi, \quad \forall x \varphi, \\
&\exists X \varphi, \quad \forall X \varphi, \\
&\mathfrak{R}(s), \quad s\downarrow
\end{aligned}$$

- We have modus ponens and quantifier-axioms/rules.

- It has an *intensional equality*

Explicit Mathematics

- Any term can "act as an operation"
- For two terms s and t, we can apply

$s(t)$	"operation s applied to argument t"
$t(s)$	"operation t applied to argument s"

Explicit Mathematics

- Undefinedness built in.
- Consider the natural numbers $\mathbb{N} = 0, 1, 2, \ldots$

$$x + \mathrm{solve}(x, y) = y \quad \Leftrightarrow \quad y - x = \mathrm{solve}(x, y).$$

- $3 + s \stackrel{?}{=} 2$
- -1 is not a natural number, so $\mathrm{solve}(3, 2)$ should be undefined.
- Explicit Mathematics has a statement for that:
- $\mathrm{solve}(2, 3)\downarrow \wedge \mathrm{solve}(2, 3) = 1$ but $\neg\mathrm{solve}(3, 2)\downarrow$.

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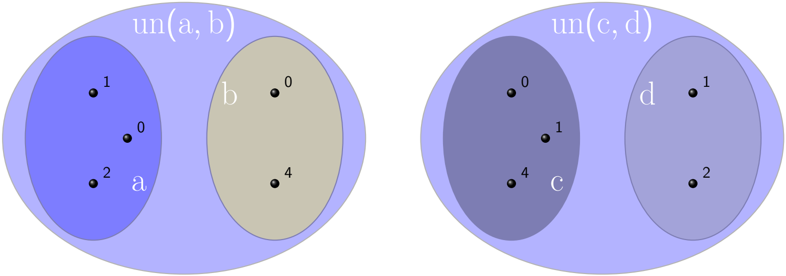
- $\mathrm{solve}(2, 3)\downarrow \wedge \mathrm{solve}(2, 3) = 1$ but $\neg\mathrm{solve}(3, 2)\downarrow$.
- We define $f(x) :\equiv x + 1$

  Then $\quad (f(1))\downarrow, \quad$ but $\quad \neg(1(f))\downarrow.$

Classes in Explicit Mathematics have names



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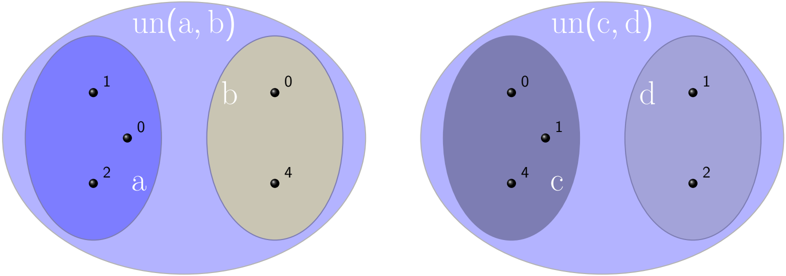

Name	Class
a	$\{0, 1, 2\}$
b	$\{0, 4\}$
c	$\{0, 1, 4\}$
d	$\{1, 2\}$
$\mathrm{un}(\mathrm{a}, \mathrm{b})$	$\{0, 1, 2, 4\}$
$\mathrm{un}(\mathrm{c}, \mathrm{d})$	$\{0, 1, 2, 4\}$

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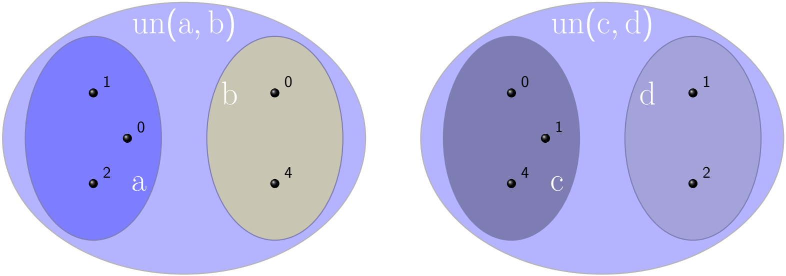

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$\mathrm{un}(\mathrm{c}, \mathrm{d})$	$\{0, 1, 2, 4\}$

x is an element of $\mathrm{un}(a, b)$
if and only if
x is an element of $\mathrm{un}(c, d)$

but

$\mathrm{un}(a, b) \neq \mathrm{un}(c, d)$

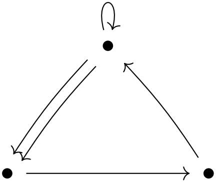
Some Notation for Explicit Mathematics

Notation

$t\downarrow$	"t is defined"
$\mathfrak{R}(a)$	"a is a name of some class"
$x \mathrel{\dot{\in}} a$	"x is an element in the class named by a."
fx	"function application $f(x)$"

Categories

- A category has *objects* and *morphism* (points and arrows)



Categories

- A category has *objects* and *morphism* (points and arrows)

$$\begin{tikzcd}
 & \bullet \arrow[loop above] \arrow[dl, shift right] \arrow[dl, shift left] & \\
\bullet \arrow[rr] & & \bullet \arrow[ul]
\end{tikzcd}$$

- with composition: for all

$$\begin{tikzcd}
 & a \arrow[dl, "f"'] & \\
b \arrow[rr, "g"'] & & c
\end{tikzcd}
\quad \text{there is some } g \circ f : \quad
\begin{tikzcd}
 & a \arrow[dl, "f"'] \arrow[dr, "g\circ f"] & \\
b \arrow[rr, "g"'] & & c
\end{tikzcd}$$

Two Laws:

- Associativity of arrows:

$$\begin{tikzcd}
\bullet \arrow[rr, "g\circ f"] & & \bullet \arrow[r, "h"] & \bullet \\
& & \Vert & \\
\bullet \arrow[r, "f"] & \bullet \arrow[r, "g"] & \bullet \arrow[r, "h"] & \bullet \\
& & \Vert & \\
\bullet \arrow[r, "f"] & \bullet \arrow[rr, "h\circ g"] & & \bullet
\end{tikzcd}$$

- Generally written as

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] \arrow[rrr, bend right, "h\circ(g\circ f) = h\circ g\circ f = (h\circ g)\circ f"'] & \bullet \arrow[r, "g"] \arrow[rr, bend right, "h\circ g"] & \bullet \arrow[r, "h"] & \bullet
\end{tikzcd}$$

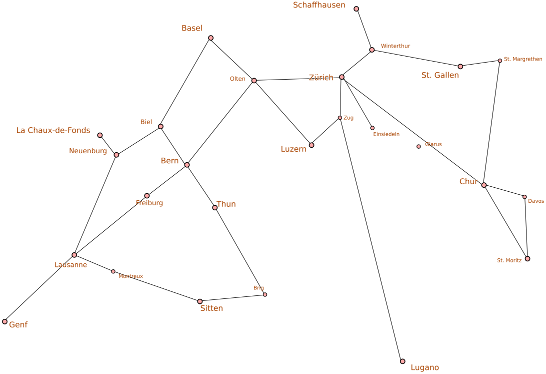
Two Laws:

- Identity: All objects have an identity arrow: $\begin{tikzcd} \bullet\, a \arrow[loop left, "\mathrm{id}(a)"] \end{tikzcd}$

  For all arrows $f : a \to b$ we require

$$\begin{tikzcd} a \arrow[loop left, "\mathrm{id}(a)"] \arrow[r, "f"] & b \end{tikzcd} = \begin{tikzcd} a \arrow[r, "f"] & b \end{tikzcd} = \begin{tikzcd} a \arrow[r, "f"] & b \arrow[loop right, "\mathrm{id}(b)"] \end{tikzcd}$$

Example 1: Traveling by train



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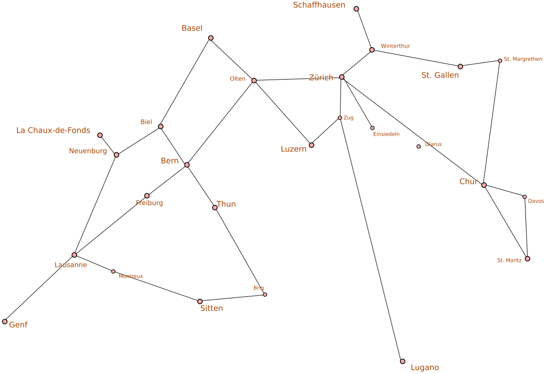
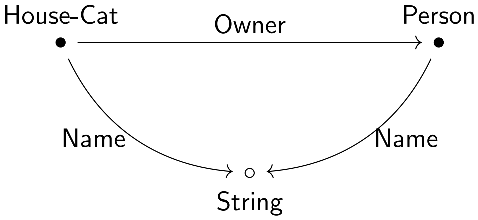


- Objects: Cities
- Morphisms: Traveling by train (in both directions)
- Composition: Taking consecutive train lines
- Identity: Staying in a city
- (Bern $\rightarrow$ Lausanne $\rightarrow$ Sion $\rightarrow$ Thun $\rightarrow$ Bern) $\neq \mathrm{id}$(Bern).

Example 2: Database design



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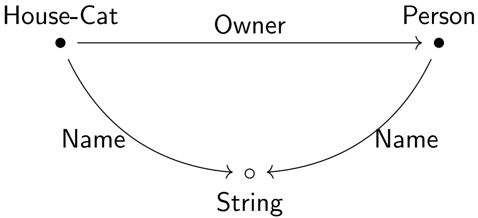

House-Cat	Name	Owner
C1	Schrödinger	P7
2	Snowball II	P8

Person	Name
P7	Alice
P8	Bob

String
Alice
Bob
Snowball II
Schrödinger

Example 3: Categories of natural numbers

- Discrete:

$$\begin{tikzcd}
\bullet\, 0 \arrow[loop above, "\mathrm{id}(0)"] & \bullet\, 1 \arrow[loop above, "\mathrm{id}(1)"] & \bullet\, 2 \arrow[loop above, "\mathrm{id}(2)"] & \bullet\, 3 \arrow[loop above, "\mathrm{id}(3)"] & \cdots
\end{tikzcd}$$

- Totally Ordered:

$$\begin{tikzcd}
\bullet\, 0 \arrow[loop below, "\mathrm{id}(0)"] & \bullet\, 1 \arrow[l, "\leq"'] \arrow[loop below, "\mathrm{id}(1)"] & \bullet\, 2 \arrow[l, "\leq"'] \arrow[loop below, "\mathrm{id}(2)"] & \bullet\, 3 \arrow[l, "\leq"'] \arrow[loop below, "\mathrm{id}(3)"] & \cdots \arrow[l, "\leq"']
\end{tikzcd}$$

Example 4: “Traditional” Categories

Name	Objects	Morphisms
Grp	Groups	Group homomorphisms
Rng	Commutative Rings	Ring homomorphisms
Top	Topological spaces	Continuous maps
hTop	Topological spaces	Homotopy classes of cont. maps
Vect	Vector spaces	Linear maps
Hilb	Hilbert spaces	Bounded linear operators
	Lattice elements	$\leq$
	$\bullet$	Group elements

Idea of Category Theory

- Study of the structural properties of a subject.
- Relations between objects are more important than how a single object is defined.
- Tells us what to look for when studying a new field.
  (Formal definition of products / sums / quotients /etc.)
- It lets us transfer knowledge from one field to another.

Two Styles of Category Theory

- *Cat*-Category Theory
  - Diagram chasing
  - Equational Reasoning
  - Adjunctions in unit/counit formulation
- *Set*-Category Theory
  - Representability
  - "Proof by Yoneda"
  - Adjunctions in the form of $\mathbf{Hom}(FA, B) \cong \mathbf{Hom}(A, UB)$.

Category of Sets

- Category of sets and functions
- Very good Properties
- Extremely well-studied (Set Theory)
- Other categories are studied in relation to the category of sets.

“Category of sets” in Explicit Mathematics

- Explicit Mathematics does not have sets it has classes
- Question: How many properties of sets can we get?

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- Explicit Mathematics does not have sets it has classes
- Question: How many properties of sets can we get?
- Short answer: Surprisingly many!

The category **EC** (Elementary Comprehension)

- The most "natural" category in Explicit Mathematics
- Objects: Classes
- Morphisms: (total) operations between classes
  such that $(f =_{\mathbf{EC}} g) : a \to b$ if and only if $(\forall x \mathrel{\dot\in} a)(fx = gx)$.

The category **EC** (Elementary Comprehension)

- Main problem: **EC** does not have “function spaces”
- It is not *cartesian closed*:

The category **EC** (Elementary Comprehension)

- Main problem: **EC** does not have "function spaces"
- It is not *cartesian closed*:
- For every operation

  $$g : a \times b \to c$$

  there should be exactly one operation

  $$\hat{g} : a \to c^b$$

  into the *Exponential Object* ("function space") of operations from the class b to the class c.
- There are "too many" terms representing the same operation.

The Category **EC**

- **EC** has all finite limits:
  - It has all finite products
  - and all preimages:

    For $f : x \to y$ and $b \mathrel{\dot{\subset}} y$ there always exists a preimage class $f^{-1}\{b\}$.

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- **EC** has all binary coproducts (disjoint unions)

- Preimages and coproducts interact well. (**EC** is an *extensive* category)

Extensiveness

- A set-like property of disjoint unions:
- Maps into disjoint unions should be determined by two maps into its parts:

$$
\begin{tikzcd}
\bullet \arrow[d, "h"] & & \bullet \arrow[d, "f"] & \bullet \arrow[d, "g"] \\
a \uplus b & \leftrightsquigarrow & a & b
\end{tikzcd}
$$

- If a category satisfies this, it is called *extensive*.
- A category which is exact and extensive is called a *pretopos*.

Bishop sets

A set is not an entity which has an ideal existence. A set exists only when it has been defined. To define a set we prescribe, at least implicitly, what we (the constructing intelligence) must do in order to construct an element of the set, and what we must do to show that two elements of the set are equal. (Bishop)

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Two Interpretations for equality:

- We "must *show* something."
  It is enough to prove a proposition:
  implicit Bishop sets
- We have to *constuct* something. This leads to the defintion of:
  explicit Bishop sets.

The category **ECB**

- *Implicit Bishop sets*
- The "obvious" way to fix the problem of cartesian closure in **EC**.
- Each object is now represented by pair $\langle z, r\rangle$ of classes
- such that $r \subseteq z \times z$ is an equivalence relation on z.

The category **ECB**

- *Implicit Bishop sets*
- The "obvious" way to fix the problem of cartesian closure in **EC**.
- Each object is now represented by pair $\langle z, r\rangle$ of classes
- such that $r \subseteq z \times z$ is an equivalence relation on z.
- Morphisms $f : \langle z, r\rangle \to \langle y, s\rangle$ are operations between the classes which respect the given equivalence relations:

$$(\forall a \mathrel{\dot\in} z)(f(a) \mathrel{\dot\in} y) \wedge (b \approx_r a \Rightarrow f(b) \approx_s f(a))$$

The category **ECB**

- **ECB** is (Locally) cartesian closed
- Quotients are constructed as equivalence relations on classes.
- **ECB** is a *regular category*
- finitely complete (has all finite limits)
- disjoint & stable binary coproducts

Regularity

- Given any morphism $f : a \to b$ we'd like to be able to form the quotient "$a \diagup (f(a_0) = f(a_1))$."

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- Equivalent formulations: Any morphism $f : a \to b$ has a pullback-stable factorization into a regular epimorphism followed by a monomorphism.

- "The category has a strong-enough notion of images"

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, two heads, "p"'] & b \\
im(f) \arrow[ur, tail, "\tilde{f}"'] &
\end{tikzcd}$$

Regularity: Why do we care?

- Regular categories have an internal language with existence $\exists$ and conjunction $\wedge$.
- They have a calculus (in fact a category) of *relations*.

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- Regular categories have an internal language with existence $\exists$ and conjunction $\wedge$.
- They have a calculus (in fact a category) of *relations*.
- Relations have a meet-semilattice ordering preserved under composition of relations:

$$R \leq S \Rightarrow \begin{cases} R \circ T \leq S \circ T \\ Q \circ R \leq Q \circ S \end{cases}$$

- Every relation $R(x, y)$ have an opposite relation $R^{\circ}(y, x)$.
- Relations have binary intersections:

$$(R \cap S)(x, y) \qquad \text{if and only if} \qquad R(x, y) \text{ and } S(x, y)$$

Regularity: Why do we care?

Regular categories have arrows that "work like in set theory"

- Every arrow $f : a \to b$ has a graph $G_f(a, b)$.
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  - total:
    $$\Delta_a \leq (F^\circ \circ F)$$
    (For all b_0 there exists a_0 such that $F(a_0, b_0)$)
  - functional:
    $$(F \circ F^\circ) \leq \Delta_b$$
    (For all $F(a_0, b)$, $F(a_1, b)$ we have $a_0 = a_1$)

  There exists a unique arrow $f_F : a \to b$

Exactness

- In set theory (take ZFC):
  If $R \subset X \times X$ is a *binary equivalence relation* on X we can form the quotient set $X \diagup R$.
- In a regular category, we can do the same thing in those cases where R is generated by a graph.
- We would like this property for all equivalence relations.
- If this holds, we call this an *exact category*.

The category **ECB**

- The category of implicit Bishop sets is *exact* if we allow a choice principle:

$$(AC_V) \qquad \forall x \exists y A[x, y] \Rightarrow \exists f \forall x A[x, f(x)]$$

We need a single instance of $A[x, y]$:

$$(\mathfrak{R}(x) \wedge \exists z (z \mathrel{\dot{\in}} x)) \Rightarrow y \mathrel{\dot{\in}} x$$

Finite cocompleteness

- We would like to have the equivalence relation (and quotient) generated by
$$(a \uplus b) \big/ (f(c) \approx g(c))$$
for any two arrows $f : c \to a$, $g : c \to b$.

Finite cocompleteness

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$$(a \uplus b) / (f(c) \approx g(c))$$
for any two arrows $f : c \to a$, $g : c \to b$.
- May seem arbitrary, but results in combination with all disjoint unions an a very strong property. The category has all finite colimits. "it is finitely cocomplete.")

The category **ECB**

Two results about **ECB**:

Theorem

With Classical Logic:

ECB *is a finitely complete, finitely cocomplete, locally cartesian closed, extensive category with a natural numbers object.*

Theorem

With the axiom (AC_V) :

ECB *is a locally cartesian closed arithmetic pretopos.*

The category **ECB**

Theorem

With **ECB** *as the category of sets, the Yoneda Lemma holds.*

Corollary

For "nice" categories $\mathcal{C}$ it is enough to consider maps into an object to recover that object.

Let c, d be two objects of $\mathcal{C}$. If there is a natural isomorphism

$$\mathcal{C}(a, c) \cong \mathcal{C}(a, d),$$

then

$$c \cong d.$$

The category $\mathbf{EC}_{\mathrm{ex}}$

- Classical logic and choice are both rather strong demands for our system.
- Want a way to get exactness in a way which is (more) constructive.
- *Exact Completion* of a finitely complete category $\mathcal{C}_{\mathrm{ex}}$ (also known as $\mathcal{C}_{ex/lex}$).

The category $\mathbf{EC}_{\mathrm{ex}}$

- *explicit Bishop sets*
- Similar to **ECB** but has *pseudo-equivalence relations* as objects instead of equivalence relations.
- $x \approx y$ is no more represented as a pair $\langle x, y \rangle$ but as an arbitrary element p of some class of proof-objects which witness the equivalence $p : x \approx y$.
- arrows are now two operations $\langle f, g \rangle : a \to b$
  A map of elements and a map of proof-objects

$$p : (x \approx_a y) \quad \Rightarrow \quad g(p) : (f(x) \approx_b f(y)).$$

Equivalence Relations vs. Pseudo-equivalence Relations

$$\begin{tikzcd}[column sep=small] & \{\langle 0,0\rangle,\langle 1,1\rangle,\langle 0,1\rangle,\langle 1,0\rangle\} \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\ \{0,1\} & & \{0,1\} \end{tikzcd}$$

Is an equivalence relation setting $0 \approx 1$.

$$\begin{tikzcd}[column sep=small] & \left\{\begin{matrix}\langle 0,\blacktriangle,0\rangle,\langle 1,\blacktriangle,1\rangle,\langle 0,\blacktriangle,1\rangle,\langle 1,\blacktriangle,0\rangle, \\ \langle 0,\blacktriangledown,0\rangle,\langle 1,\blacktriangledown,1\rangle,\langle 0,\blacktriangledown,1\rangle,\langle 1,\blacktriangledown,0\rangle\end{matrix}\right\} \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_2"] & \\ \{0,1\} & & \{0,1\} \end{tikzcd}$$

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The category $\mathbf{EC}_{\mathrm{ex}}$

Theorem

(Without any extra assumptions)

- *In Explicit Mathematics:* $\mathbf{EC}_{ex}$ *is exact.*
- *From outside:* $\mathbf{EC}_{ex}$ *is extensive because* $\mathbf{EC}$ *is. (Menni 2000)*
- *From outside:* $\mathbf{EC}_{ex}$ *is a pretopos.*

Conjecture

The proof for extensiveness can be internalized.

It has a "straightforward but tedious" proof.

$\mathbf{EC}_{\mathrm{ex}}$ is "too well-behaved"

- The definitions of categories, functors and natural transformations "live in **ECB**."
- The Yoneda Lemma is not provable.

$\mathbf{EC}_{\mathrm{ex}}$ is "too well-behaved"

- The definitions of categories, functors and natural transformations "live in **ECB**."
- The Yoneda Lemma is not provable.
- Future direction: Categories enriched in explicit Bishop sets.

Universes

- A collection closed under all interesting properties.
- This depends strongly on what one means by “interesting”.
- Category Theory and Explicit Mathematics disagree on this.

Universes in Explicit Mathematics

- A class which contains only names.
- Closed under constructions of names.
- Let u be a universe in that sense:

  $$\text{if } a \mathrel{\dot{\in}} u \text{ and } b \mathrel{\dot{\in}} u \text{ then } \mathtt{un}(a, b) \mathrel{\dot{\in}} u$$

  If two names are contained in u then also the name of the union directly constructed from them.
- Advantage: Very easy definition.
- Disadvantage: Not possible to close under *all* names of a class.

Universes in Category Theory

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  - Left-, and right-adjoint to the pullback-functor: "Closure under existence,- and forall-quantifier in the internal language"

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  - Should be nontrivial (Empty universes are of course closed under everything.)
- Closure under isomorphisms is inconsistent with universes as classes.
- A categorical universe is described by a formula $\mathfrak{CU}[\cdot,\cdot]$.

$$\mathfrak{CU}[u, f : a \to b] :\Leftrightarrow f : a \to b \text{ is in the universe } u.$$

Universes: An Overview

EC Universe
$\mathfrak{U}_{\mathbf{EC}}$ (A class in Explicit Math.)

ECB Universe $\mathfrak{U}_{\mathbf{ECB}}$
Class of implicit Bishop sets constructed from $\mathfrak{U}_{\mathbf{EC}}$.

"Locally small" category
$\mathcal{C} \dot{\in} \mathfrak{U}_{\mathbf{EC}}$
with $\mathfrak{U}_{\mathbf{EC}} \dot{\in} \mathfrak{U}_{\mathbf{ECB}}$

Category theoretic universe $\mathfrak{CU}[\mathfrak{U}_{\mathbf{ECB}}, \cdot]$

Yoneda Lemma holds for $\mathcal{C}$

Categorical Universes in **ECB**

It is possible to interpret $\mathfrak{CU}$ in implicit Bishop sets (**ECB**).

- Suppose we are given a universe u in the sense of Explicit Mathematics.
- We define $\mathfrak{CU}[u, f : a \to b]$ to mean

  "f is small (w.r.t. u) if and only if all its preimages are small."

In a bit more details:

Definition

The morphism $f : a \to b$ of implicit Bishop sets is in the universe u
if and only if
for all $y \,\dot{\in}\, b$, the universe u contains the name of an isomorphic copy of the preimage $f^{-1}\{y\}$.

Categorical Universes in **ECB**

Two notes on the Construction:

- The universe has a weakly classifying morphism.
  All arrows arise as the pullback along some (non-unique!) morphism of the weakly classifying one.

- My construction requires the Join axiom for the proof of closure under left-, and right-adjoints to the pullback-functor.

Thank You

A Categorical Universe

Definition

Let $\mathcal{C}$ be a locally cartesian closed category, *el* be some morphism in $\mathcal{C}$ and $\mathcal{S}[x]$ be a formula. We call $\mathcal{S}$ a universe in $\mathcal{C}$ if the following axioms hold.

$$(U1) \quad Mor(a) \wedge Mor(f) \wedge \mathcal{S}[a] \Rightarrow (PB[a, f, pr_0, pr_1] \Rightarrow \mathcal{S}[pr_0])$$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "h"] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \mathcal{S}[h] \Rightarrow \mathcal{S}[g]$$

$$(UW2) \quad Mor(f, g) \wedge ISO[f, g] \Rightarrow \mathcal{S}[f] \wedge \mathcal{S}[g]$$

$$(U3) \quad f : b \to c \wedge g : a \to b \wedge \mathcal{S}[f] \wedge \mathcal{S}[g] \Rightarrow \mathcal{S}[\Sigma_f g]$$

$$(U4) \quad f : a \to i \wedge g : b \to a \wedge \mathcal{S}[f] \wedge \mathcal{S}[g] \Rightarrow \mathcal{S}[\Pi_f g]$$

$$(U5) \quad Mor(a) \wedge \mathcal{S}[a] \Rightarrow \exists f, pr_1(f : cod(a) \to cod(el) \wedge PB[f, el, a, pr_1])$$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "a"'] \arrow[dr, phantom, "\lrcorner", very near start] & e \arrow[d, "el"] \\ \bullet \arrow[r, "\exists f"] & u \end{tikzcd}$$

A Categorical Universe in **ECB**

Definition (Categorical Universe of Bishop Sets)

Now we say a morphism is part of the categorical universe ($\mathfrak{CU}$) relative to u if the following formula is true.

$$
\begin{aligned}
\mathfrak{CU}[f,u] :\equiv \exists h, h^{-1}\exists g(\forall x \,\dot{\in}\, cod(f))(g[x] \,\dot{\in}\, u \\
&\wedge (\forall y \,\dot{\in}\, cod(f))(x \approx_{cod(f)} y \Rightarrow \forall z(z \,\dot{\in}\, g[x] \Leftrightarrow z \,\dot{\in}\, g[y])) \\
&(h[x] : f^{-1}\{x\} \to g[x] \wedge (\forall y \,\dot{\in}\, cod(f)) \\
&(x \approx_{cod(f)} y \Rightarrow (\forall z \,\dot{\in}\, f^{-1}\{x\})(\overline{(h[x])}z \approx_{g[x]} \overline{(h[y])}z))) \\
&\wedge h^{-1}[x] : g[x] \to f^{-1}\{x\} \wedge (\forall y \,\dot{\in}\, cod(f)) \\
&(x \approx_{cod(f)} y \Rightarrow \\
&(\forall z \,\dot{\in}\, g[x])(\overline{(h^{-1}[x])}z \approx_{(f^{-1}\{x\})} \overline{(h^{-1}[y])}z))) \\
&\wedge iso(h[x], h^{-1}[x]))
\end{aligned}
$$

Exactness

Definition

A finitely complete category is called exact, if every kernel pair has a coequalizer, regular epis are stable under pullback and every congruence is a kernel pair.

- A Congruence is an internal equivalence relation. $R \xrightarrow{\langle r_0, r_1 \rangle} X \times X$ satisfying reflexivity, symmetry and transitivity.

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- $$\begin{tikzcd} R \arrow[r, "r_1"] \arrow[d, "r_0"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & X \end{tikzcd}$$ is a pullback diagram for some f.

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- $R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X \twoheadrightarrow Im(f)$ is a coequalizer diagram (quotient.)

Congruence

Congruences can be characterized by a fife morphisms
$\langle r_0, r_1, \mathit{refl}, \mathit{sym}, \mathit{tr} \rangle$: such that $R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X$ are jointly monic and

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$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_0"] & \\ X & X \arrow[l, "id_X"] \arrow[r, "id_X"'] \arrow[u, "refl"] & X \end{tikzcd}$$

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X & X \arrow[u, "refl"] \arrow[l, "id_X"] \arrow[r, "id_X"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_0"] & \\
X & R \arrow[u, "sym"] \arrow[l, "r_0"] \arrow[r, "r_1"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & R * R \arrow[dlll, "tr"'] \arrow[dll, "r_0^*"] \arrow[drr, "r_1^*"'] \arrow[drrr, "tr"] & & & \\
R \arrow[d, "r_0"'] & R \arrow[dl, "r_0"] \arrow[drr, "r_1"] & & & & R \arrow[dll, "r_0"'] \arrow[dr, "r_1"'] & R \arrow[d, "r_1"] \\
X & & & X & & & X
\end{tikzcd}$$

Congruence

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$\langle r_0, r_1, \mathit{refl}, \mathit{sym}, \mathit{tr}\rangle$: such that $R \overset{r_0}{\underset{r_1}{\rightrightarrows}} X$ are jointly monic and

$$
\begin{tikzcd}
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_0"] & \\
X & X \arrow[l, "id_X"] \arrow[r, "id_X"'] \arrow[u, "refl"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & R*R \arrow[dll, "tr"'] \arrow[dl, "r_0^*"] \arrow[dr, "r_1^*"'] \arrow[drr, "tr"] \arrow[d, phantom, "\vee"] & & \\
R \arrow[d, "r_0"'] & R \arrow[dl, "r_0"] \arrow[dr, "r_1"] & & R \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & R \arrow[d, "r_1"] \\
X & & X & & X
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_0"] & \\
X & R \arrow[l, "r_0"] \arrow[r, "r_1"'] \arrow[u, "sym"] & X
\end{tikzcd}
$$

$$\vdash_{\{x:X\}} R(x,x)$$

$$R(x,y) \vdash_{\{x:X,y:X\}} R(y,x)$$

$$R(x,y) \wedge R(y,z) \vdash_{\{x:X,y:X,z:X\}} R(x,z)$$

Notation

- $R \rightrightarrows X$ is a pseudo-equivalence relation constructed from the morphisms $\langle r_0, r_1, \mathit{refl}_R, \mathit{sym}_R, \mathit{tr}_R \rangle$ with $r_0, r_1 : R \to X$
- We write $z : a\ R\ b$ for an element $z \in R$ with $r_0(z) = a$ and $r_1(z) = b$.

Exact Completion $\mathbf{EC}_{\mathrm{ex}}$

Objects of the exact completion are pseudo-equivalence relations. $\langle r_0, r_1, \mathit{refl}, \mathit{sym}, \mathit{tr} \rangle$ with

- $\vdash_{\{x:X\}} r_0(\mathit{refl}(x)) = x \wedge r_1(\mathit{refl}(x)) = x$
- $\vdash_{\{r:R\}} r_0(\mathit{sym}(r)) = r_1(r) \wedge r_1(\mathit{sym}(r)) = r_0(r)$
- $\vdash_{\{p:R*R\}} r_0(\mathit{tr}(p)) = r_0(r_0^*(p)) \wedge r_1(\mathit{tr}(p)) = r_1(r_1^*(p))$

Without the requirement that $R \xrightarrow{\langle r_0, r_1 \rangle} X \times X$ is a mono.

Exact Completion $\mathbf{EC}_{\mathrm{ex}}$

Morphisms are given by a pair of maps $\langle f, f' \rangle$ in **EC** with $s_i \circ f' = f \circ r_i$:

$$\begin{tikzcd}
R \arrow[r, "f'"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & S \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

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subject to the equivalence relation
$\langle f, f' \rangle = \langle g, g' \rangle \Leftrightarrow (\exists \gamma : X \to S)(s_0 \circ \gamma = f \wedge s_1 \circ \gamma = g)$

$$\begin{tikzcd} & S \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] \\ X \arrow[ur, "\exists\gamma"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

$$\gamma(x) : f(x) \; S \; g(x)$$

Functor $\Gamma : \mathcal{C} \to \mathcal{C}_{\text{ex}}$

- $\gamma_o(a) :\equiv \langle id(a), id(a), id(a), id(a), id(a) \rangle$
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- $\gamma_m(f : a \to b) :\equiv \langle f, f \rangle : \gamma_o(a) \to \gamma_o(a)$

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, shift right, "id(a)"'] \arrow[d, shift left, "id(a)"] & b \arrow[d, shift right, "id(b)"'] \arrow[d, shift left, "id(b)"] \\
a \arrow[r, "f"'] & b
\end{tikzcd}$$

“Universal Property” (???)

Probably: Equivalence of categories $\mathrm{Exact}(\mathcal{C}_{\mathrm{ex}}, \mathcal{D}) \sim \mathrm{Lex}(\mathcal{C}, \mathcal{D})$. for any exact category $\mathcal{D}$.

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\Gamma(r) \arrow[r, shift left, "\Gamma(r_0)"] \arrow[r, shift right, "\Gamma(r_1)"'] \arrow[d, "\Gamma(f')"] & \Gamma(x) \arrow[r, "\langle id(x), refl_{rx} \rangle"] \arrow[d, "\Gamma(f)"] & (r \Rightarrow x) \arrow[d, "\langle f, f' \rangle"] \\
\Gamma(s) \arrow[r, shift left, "\Gamma(s_0)"] \arrow[r, shift right, "\Gamma(s_1)"'] & \Gamma(y) \arrow[r, "\langle id(x), refl_{rx} \rangle"'] & (s \Rightarrow y)
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If $G : \mathcal{C} \to \mathcal{D}$ is a functor which preserves finite limits, then we can construct

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If $G : \mathcal{C} \to \mathcal{D}$ is a functor which preserves finite limits, then we can construct

$$
\begin{tikzcd}
G(r) \arrow[r, shift left, "G(r_0)"] \arrow[r, shift right, "G(r_1)"'] \arrow[d, "G(f')"] & G(x) \arrow[r, "coeq_{\mathcal{D}}(G(r_0), G(r_1))"] \arrow[d, "G(f)"] & cod(coeq_{\mathcal{D}}(G(r_0), G(r_1))) \arrow[d, dotted, "cext_{\mathcal{D}}(coeq_{\mathcal{D}}(G(s_0), G(s_1)) \circ G(f))"] \\
G(s) \arrow[r, shift left, "G(s_0)"] \arrow[r, shift right, "G(s_1)"'] & G(y) \arrow[r, "coeq_{\mathcal{D}}(G(s_0), G(s_1))"'] & cod(coeq_{\mathcal{D}}(G(s_0), G(s_1)))
\end{tikzcd}
$$