# Trees Describing Topological Weihrauch Degrees of Multivalued Functions

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# Overview

- Motivation
- Wadge/Strong Weihrauch/Weihrauch reducibility: Several versions
- Better-quasi-orders, labeled forests
- Forests describing discontinuities of (multi-valued) functions
- Combinatorial description of an initial segment of the topological Wadge/strong Weihrauch/Weihrauch degrees of functions with range in a better-quasi-order and of multi-valued functions with finite discrete range

# Motivation

## Goal

Describe the discontinuities appearing in computation problems in a combinatorial way that allows us to compare them easily.

## Why?

Discontinuities cause problems when computing real number functions.

- Discontinuities in
  - numerical computation: *instabilities*.
  - computational geometry: *degenerate configurations*.
- In Computable Analysis (Turing machine model, computing with "finite" (rational, dyadic) approximations):

  *Computable functions are continuous*.

  Degrees of discontinuity are *topological degrees of noncomputability*.

# Wadge Reducibility

For subsets $A, B \subseteq \mathbb{B} := \mathbb{N}^{\mathbb{N}}$:

$$\begin{aligned} A \leq_0 B \quad & :\Longleftrightarrow \quad (\exists \text{ cont. } I : \mathbb{B} \to \mathbb{B})\ A = I^{-1}(B), \\ & \Longleftrightarrow \quad (\exists \text{ cont. } I : \mathbb{B} \to \mathbb{B})\ (\forall x \in \mathbb{B})\ \mathrm{cf}_A(x) = \mathrm{cf}_B(I(x)), \end{aligned}$$

*$A$ is Wadge reducible to $B$*.

Wadge (1984) characterized the Wadge degrees of all Borel subsets of $\mathbb{B}$: an almost linear structure.

# Relations/Multivalued Functions

Let $X, Y$ be sets.
Often a computational problem can be formulated as a relation (multivalued function) $R \subseteq X \times Y$:

> *Given some $x \in X$,*
> *compute a $y \in Y$ with the property $(x, y) \in R$.*

For a relation $R \subseteq X \times Y$ we define

$$\begin{aligned} \mathrm{dom}(R) &:= \{x \in X \ : \ (\exists y \in Y)\,(x, y) \in R\}, \\ R[M] &:= \{y \in Y \ : \ (\exists x \in M)\,(x, y) \in R\}, \end{aligned}$$

for $M \subseteq X$. And $R[x] := R[\{x\}]$. We define the composition $S \circ R$ for relations $R \subseteq X \times Y$, $S \subseteq Y \times Z$ as follows. For all $x \in X$

$$(S \circ R)[x] := \begin{cases} \emptyset & \text{if } R[x] \not\subseteq \mathrm{dom}(S), \\ S[R[x]] & \text{if } R[x] \subseteq \mathrm{dom}(S). \end{cases}$$

# Continuous Reductions Between Relations: Naive Version

Let $X, Y, X', Y'$ be topological spaces,
$R \subseteq X \times Y$, $R' \subseteq X' \times Y'$ be relations.

## Definition

$R \leq_0 R' :\iff (\exists \text{ cont. } I)\ (\forall x \in \text{dom}R)$ $R[x] \supseteq R'[I(x)] \supsetneq \emptyset$ (naive Wadge reducibility)

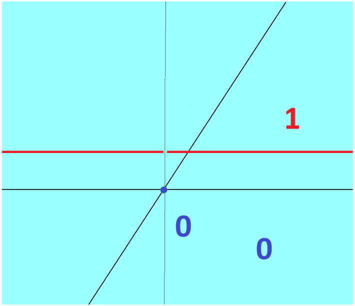
$R \leq_1 R' :\iff (\exists \text{ cont. } I, O)\ (\forall x \in \text{dom}R)$ $R[x] \supseteq O[R'[I(x)]] \supsetneq \emptyset$ (naive strong Weihrauch reducibility)

$R \leq_2 R' :\iff (\exists \text{ cont. } I, O)\ (\forall x \in \text{dom}R)$ $R[x] \supseteq O[x, R'[I(x)]] \supsetneq \emptyset$ (naive Weihrauch reducibility)

# An Example

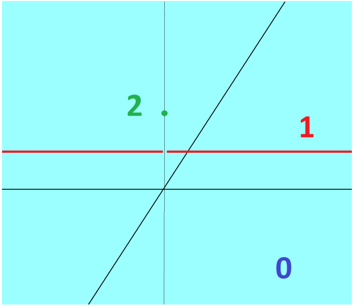
$f : \mathbb{R}^2 \to \{0, 1, 2\}$

$$f(x) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ 1 & \text{if } y = 0 \wedge x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$




$g : \mathbb{R}^2 \to \{0, 1, 2\}$

$$g(x) = \begin{cases} 2 & \text{if } (x, y) = (0, 0) \\ 1 & \text{if } y = 0 \wedge x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$




$f \leq_1 g$,

$g \not\leq_2 f$.

# Caveat Concerning the Naive Versions of Continuous Reductions on Arbitrary Spaces

Often connectedness properties of the topological spaces preclude the existence of continuous reduction functions, and this does not seem to make sense from a computational point of view.

Suggestion:

- Do not $\leq_i$-compare multivalued functions $R, R'$ directly,
- but $\leq_i$-compare them *relative to admissible representations*!

# Type-two Theory of Effectivity

Let $X$ be a set. A *representation of $X$* is a surjective function $\delta_X :\subseteq \mathbb{B} \to X$.

$$\begin{array}{ccc} & R & \\ X & \rightrightarrows & Y \\ \delta_X \uparrow & & \uparrow \delta_Y \\ \mathbb{B} & \longrightarrow & \mathbb{B} \\ & \text{realizer} & \end{array}$$

A function $F :\subseteq \mathbb{B} \to \mathbb{B}$ is called a $(\delta_X, \delta_Y)$*-realizer* of a relation $R \subseteq X \times Y$ if

$$(\forall p \in \mathrm{dom}(R \circ \delta_X)) \quad \delta_Y(F(p)) \in R[\delta_X(p)].$$

In the following we shall only consider *admissible* (i.e. topologically well-behaved) representations (introduced by Kreitz and Weihrauch for countably based $T_0$-spaces, extended by Schröder to $\text{qcb}_0$-spaces).
A relation $R \subseteq X \times Y$ is called *relatively continuous* if there exists a continuous realizer for it.

# Continuous Reductions between Relations: Full Version

Let $X, Y, X', Y'$ be qcb$_0$-spaces, $R \subseteq X \times Y$, $R' \subseteq X' \times Y'$ be relations.

## Definition

$R \leq_{\mathrm{Wa}} R' :\iff R \circ \delta_X \leq_0 R' \circ \delta_{X'}$

$\iff (\exists$ cont. $I)\,(\forall$ realizers $F'$ of $R')$ $F' \circ I$ is a realizer of $R$ (Wadge reducibility)

$R \leq_{\mathrm{sW}} R' :\iff \delta_Y^{-1} \circ R \circ \delta_X \leq_1 (\delta_{Y'}^{-1}) \circ R' \circ \delta_{X'}$

$\iff (\exists$ cont. $I, O)\,(\forall$ realizers $F'$ of $R')$ $O \circ F' \circ I$ is a realizer of $R$ (strong Weihrauch reducibility)

$R \leq_{\mathrm{W}} R' :\iff \delta_Y^{-1} \circ R \circ \delta_X \leq_2 (\delta_{Y'}^{-1}) \circ R' \circ \delta_{X'}$

$\iff (\exists$ cont. $I, O)\,(\forall$ realizers $F'$ of $R')$ $O \circ (\mathrm{id}_{\mathbb{B}}, F' \circ I)$ is a realizer of $R$ (Weihrauch reducibility)

# Well-quasi-orders and Better-quasi-orders

A relation $\leq \subseteq X \times X$ is called a *quasi-order* if it is reflexive and transitive.

It is called a well-quasi-order if it is a quasi-order and for every sequence $x_0, x_1, x_2, \ldots$ there exist $i, j$ with $i < j$ and $x_i \leq x_j$.

Nash-Williams introduced a stronger notion with better closure properties: *better-quasi-orders*.

# Wadge Reducibility for Functions with Range in a Quasi-order

Let $X, X'$ be topological spaces and $(Y, \preceq)$ be a quasi-ordered set.
For functions $f :\subseteq X \to Y$ and $f' :\subseteq X' \to Y$ we define

$$f \preceq_0 f' \quad :\iff \quad \text{there exists a continuous function } I :\subseteq X \to X' \text{ such that } (\forall x \in \mathrm{dom}(f))\ f(x) \preceq f'(I(x)).$$

For sets $X, Y$ and a relation $R \subseteq X \times Y$ we define the function $F_R :\subseteq X \to \mathcal{P}(Y)_{\neq\emptyset}$ by $\mathrm{dom}(F_R) := \mathrm{dom}(R)$ and $F_R(x) := R[x]$, for $x \in \mathrm{dom}(R)$.
Consider the quasi-ordered set $(\mathcal{P}(Y)_{\neq\emptyset}, \supseteq)$.

## Lemma

*Let $Y$ be a set, let $X, X'$ be topological spaces, and let $R \subseteq X \times Y$ and $R' \subseteq X' \times Y$ be relations. TFAE:*

1. $R \leq_0 R'$,
2. $F_R \supseteq_0 F_{R'}$.

# Strong Weihrauch Reducibility and Weihrauch Reducibility for Functions with Range in a Quasi-order

Let $X, X'$ be topological spaces.
Let $(Y, \preceq)$ and $(Y', \preceq')$ be preordered sets.
For functions $f :\subseteq X \to Y$ and $f' :\subseteq X' \to Y'$ and a monotone function $o :\subseteq Y' \to Y$ with upwards closed domain we define

$$f \preceq_1^o f' \quad :\iff \quad f \preceq_0 o \circ f',$$

and

$$f \preceq_1 f' \quad :\iff \quad \text{there exists a monotone function } o :\subseteq Y' \to Y \text{ with upwards closed domain such that } f \preceq_1^o f'.$$

$f \preceq_2 f' :\iff$ there exist an equicontinuous and monotone function $o :\subseteq X \times Y' \to Y$ with upwards closed domain and a continuous function $I :\subseteq X \to X'$ such that $(\forall x \in \mathrm{dom}(f))\ f(x) \preceq o(x, f'(I(x)))$.

# Forests and Trees

- A poset is a partially ordered set $(P, \leq)$, that is, a set $P$ with a binary relation $\leq$ on it that is reflexive, transitive, and anti-symmmetric.
- A chain is a poset $(P, \leq)$ such that $x \leq y$ or $y \leq x$, for all $x, y \in P$.
- Let $\mathbb{N}^{<\omega}$ be the set of all finite strings of natural numbers, and let $\sqsubseteq$ be the prefix relation on $\mathbb{N}^{<\omega}$.
- By a forest we mean a subset $S \subseteq \mathbb{N}^{<\omega}$ such that every $\sqsubseteq$-chain is finite.
- A tree is a forest that has a $\sqsubseteq$-smallest element, that is, an element $x$ satisfying $x \sqsubseteq y$ for all $y \in P$.
  When such an element exists then it is uniquely determined, and it is called the root of the tree.
- A $Y$-labeled forest is a triple $(S, \lambda)$ consisting of a forest $S$ and a labeling function $\lambda : S \to Y$.

# The $\leq_0$-relation on Forests

## Definition

Let $(Y, \preceq)$ be a quasi-ordered set.

1. Let $F = (S, \lambda)$ and $F' = (S', \lambda')$ be labeled forests with labels in $Y$. A monotone (with respect to $\sqsubseteq$) function $h : S \to S'$ satisfying

$$(\forall s \in S)\ \lambda(s) \preceq \lambda'(h(s))$$

is called a *morphism from $F$ to $F'$*.

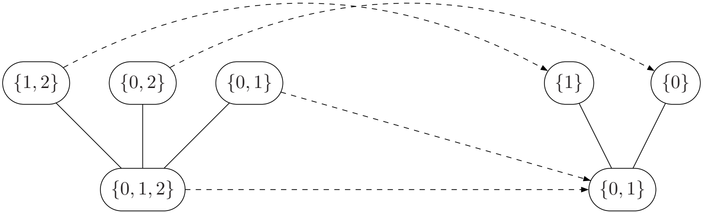
2. For any labeled forests $F$ and $F'$ with labels in $Y$, $F \preceq_0 F'$ iff there exists a morphism from $F$ to $F'$.

The relation $\preceq_0$ is reflexive and transitive.
Laver has shown: If $(Y, \preceq)$ is a better-quasi-ordered set then $\preceq_0$ is a better-quasi-order on the set of all $Y$-labeled forests.
In the following we will consider equivalence classes of forests.

# The $\leq_0$-relation on Forests

## Example

Two trees labeled with elements from $(Y, \preceq) := (\mathcal{P}(\mathbb{N}), \supseteq)$.
A morphism from the tree on the left to the tree on the right:




$\mathcal{F}_2(F_{\mathrm{LLPO}_3})$ $\mathcal{F}_2(F_{\mathrm{LLPO}_2})$

# The $\leq_1$-relation on Forests

Let $(Y \preceq), (Y', \preceq')$ be quasi-ordered sets.

## Definition

1. Let $o :\subseteq Y' \to Y$ be a monotone function with upwards closed domain. For any labeled forests $F = (T, \lambda) \in \mathrm{Forests}(Y)$ and $F' = (T', \lambda') \in \mathrm{Forests}(Y')$,

$$F \preceq_1^o F' :\Longleftrightarrow$$

there exists a monotone function $h : T \to T'$ such that $(\forall t \in T)\ (\lambda'(h(t)) \in \mathrm{dom}(o)$ and $\lambda(t) \preceq o(\lambda'(h(t)))).$

2. For any labeled forests $F = (T, \lambda) \in \mathrm{Forests}(Y)$ and $F' = (T', \lambda') \in \mathrm{Forests}(Y')$,

$$F \preceq_1 F' \quad :\Longleftrightarrow \quad \text{there exists a } \ldots \text{ function } o :\subseteq Y' \to Y \text{ such that } F \preceq_1^o F'.$$

# The $\leq_1$-relation on Forests

Let us call a poset *weakly bounded-complete* if every nonempty bounded subset $S$ has a supremum.

## Lemma

*Let $(Y, \preceq)$ be a weakly bounded-complete poset, and let $(Y', \preceq')$ be a quasi-ordered set. Then for $F = (T, \lambda) \in \mathrm{Forests}(Y)$ and $F' = (T', \lambda') \in \mathrm{Forests}(Y')$, TFAE:*

1. *$F \preceq_1 F'$.*
2. *There exists a monotone function $h : T \to T'$ such that, for all nonempty $M \subseteq T$, if $\lambda'[h[M]]$ is $\preceq'$-upper bounded then $\lambda[M]$ is $\preceq$-upper bounded.*

If $Y, Y'$ are finite sets and $\preceq, \preceq'$ are the equality then equivalent:

2' *There exists a monotone function $h : T \to T'$ such that, for all $t_1, t_2 \in T$, if $\lambda(t_1) \neq \lambda(t_2)$ then $\lambda'(h(t_1)) \neq \lambda'(h(t_2))$.*

# The $\leq_2$-relation on Forests

Let $(Y, \preceq)$ and $(Y', \preceq')$ be quasi-ordered sets.
We call a function $o :\subseteq T \times Y' \to Y$ *good* if

1. $(\forall t \in T)\ o(t, \cdot)$ is monotone, and its domain is upwards closed,
2. $(\forall t_1, t_2 \in T)\,(\forall y' \in Y')$, if $(t_1 \leq t_2$ and $(t_1, y') \in \mathrm{dom}(o)$ and $(t_2, y') \in \mathrm{dom}(o))$ then $o(t_1, y') = o(t_2, y')$.

## Definition

For $F = (T, \lambda) \in \mathrm{Forests}(Y)$ and $F' = (T', \lambda') \in \mathrm{Forests}(Y')$,

$F \preceq_2 F' :\Longleftrightarrow$

there exist a monotone function $h : T \to T'$ and

a good function $o :\subseteq T \times Y' \to Y$ such that

$(\forall t \in T)\ ((t, \lambda'(h(t)) \in \mathrm{dom}(o)$ and $\lambda(t) \preceq o(t, \lambda'(h(t))))).$

# The $\leq_2$-relation on Forests

## Lemma

*Let* $(Y, \preceq)$ *be a weakly bounded-complete poset, and let* $(Y', \preceq')$ *be a quasi-ordered set. Then for* $F = (T, \lambda) \in \mathrm{Forests}(Y)$ *and* $F' = (T', \lambda') \in \mathrm{Forests}(Y')$ *TFAE*

1. $F \preceq_2 F'$,
2. *there exists a monotone function* $h : T \to T'$ *such that for every subset* $M \subseteq T$ *having a smallest element, if* $\lambda'[h[M]]$ *is* $\preceq'$*-upper bounded then* $\lambda[M]$ *is* $\preceq$*-upper bounded.*

If $Y, Y'$ are finite sets and $\preceq, \preceq'$ are the equality then equivalent:

2' *There exists a monotone function* $h : T \to T'$ *such that, for all* $t_1, t_2 \in T$, *if* $t_1 \sqsubseteq t_2$ *and* $\lambda(t_1) \neq \lambda(t_2)$ *then* $\lambda'(h(t_1)) \neq \lambda'(h(t_2))$.

# Classes of Forests Describing the Discontinuities of Functions

Let $(Y, \preceq)$ be a quasi-ordered set. Let $X$ be a second countable topological space. Let $\mathcal{CR}(X, Y)$ be the set of all functions $f :\subseteq X \to Y$ with countable range. With any function $f \in \mathcal{CR}(X, Y)$ we associate the following $\equiv_0$-classes $\mathcal{F}_\alpha(f)$ of labeled forests with labels in $Y$, for any countable ordinal $\alpha$, as well as the following $\equiv_0$-classes $\mathcal{T}_\alpha(f, x)$ of trees, for any $x \in \mathrm{dom}(f)$ and any countable ordinal $\alpha$. We define recursively, for any countable ordinal $\alpha$,

$$\mathcal{F}_\alpha(f) := \sup_{\preceq_0}\{\mathcal{T}_\beta(f, x) : x \in \mathrm{dom}(f),\ \beta \in \mathrm{ORD},\ \beta < \alpha\},$$

$$\mathcal{T}_\alpha(f, x) := \mathrm{treeclass}\left(f(x), \min_{\preceq_0}\{\mathcal{F}_\alpha(f|_U) : U \subseteq X \text{ is open with } x \in U\}\right)$$

for any $x \in \mathrm{dom}(f)$.

# Classes of Forests Describing the Discontinuities of Functions: $\mathcal{F}_0(f)$ and $\mathcal{T}_0(f,x)$

$$\begin{aligned}
\mathcal{F}_\alpha(f) &:= \sup_{\preceq_0}\{\mathcal{T}_\beta(f,x) \;:\; x \in \mathrm{dom}(f),\ \beta \in \mathrm{ORD},\ \beta < \alpha\}, \\
\mathcal{T}_\alpha(f,x) &:= \mathrm{treeclass}\left(f(x), \min_{\preceq_0}\{\mathcal{F}_\alpha(f|_U) \;:\; U \subseteq X \text{ is open with } x \in U\}\right) \\
&\quad\ \text{for any } x \in \mathrm{dom}(f).
\end{aligned}$$

## Lemma

1. *The forest class $\mathcal{F}_0(f)$ is always defined.*
   *It is the $\equiv_0$-class of the empty forest.*
2. *For all $x \in \mathrm{dom}(f)$ the tree class $\mathcal{T}_0(f,x)$ is defined.*
   *It is the $\equiv_0$-class of the tree consisting of only one element, its root, labeled with $f(x)$.*

# Classes of Forests Describing the Discontinuities of Functions: $\mathcal{F}_0(f)$ and $\mathcal{T}_0(f, x)$

$$\mathcal{F}_\alpha(f) := \sup_{\preceq_0}\{\mathcal{T}_\beta(f,x) \;:\; x \in \mathrm{dom}(f),\ \beta \in \mathrm{ORD},\ \beta < \alpha\},$$

$$\mathcal{T}_\alpha(f,x) := \mathrm{treeclass}\left(f(x), \min_{\preceq_0}\{\mathcal{F}_\alpha(f|_U) \;:\; U \subseteq X \text{ is open with } x \in U\}\right)$$

for any $x \in \mathrm{dom}(f)$.

## Proposition

*Let $(Y, \preceq)$ be a bqo set. Then, for every countable ordinal $\alpha$,*

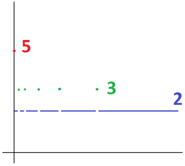
1. *$\mathcal{F}_\alpha(f)$ exists,*
2. *for every $x \in \mathrm{dom}(f)$ the tree class $\mathcal{T}_\alpha(f, x)$ exists.*

# Trees Describing the Discontinuities of Functions

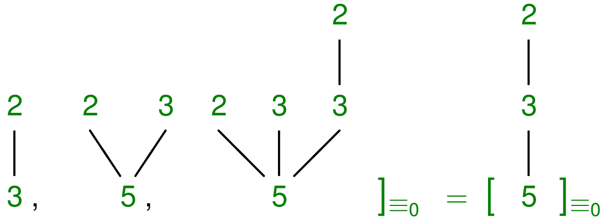
## Example

For $f : \{0, 1\}^{\mathbb{N}} \to \{2, 3, 5\}$ defined by

$$f(p) := \begin{cases} 5 & \text{if } p = 0^\omega \\ 3 & \text{if } (\exists i \in \mathbb{N})\ p = 0^i 1^\omega \\ 2 & \text{otherwise.} \end{cases}$$




$\mathcal{F}_3(f) = [\ 2\ ,\ 3\ ,\ 5\ ,$




$]_{\equiv_0} = [\ \ldots\ ]_{\equiv_0}$

$= \mathcal{F}_4(f)$

# Trees Describing the Discontinuities of Functions

## Example

Fix some natural number $n \geq 2$.

- $[n] := \{0, \ldots, n-1\}$.
- Let the bijection $\langle \cdot \rangle_n : \mathbb{B}^n \to \mathbb{B}$ be defined by

$$\langle p^{(0)}, \ldots, p^{(n-1)} \rangle_n := p_0^{(0)} \ldots p_0^{(n-1)} p_1^{(0)} \ldots p_1^{(n-1)} p_2^{(0)} \ldots p_2^{(n-1)} \ldots,$$

  for $p^{(0)}, \ldots, p^{(n-1)} \in \mathbb{B}$.

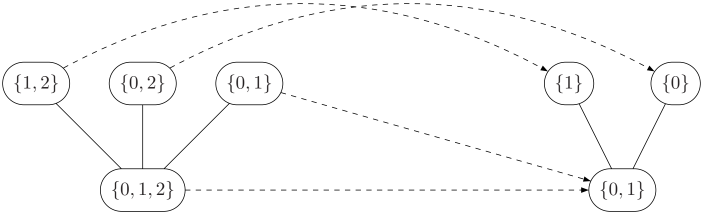
# Trees Describing the Discontinuities of Functions

## Example

For any natural number $n \geq 2$ let us define the relation $\mathrm{LLPO}_n \subseteq \mathbb{B} \times [n]$ by

$$\begin{aligned}
\mathrm{dom}(\mathrm{LLPO}_n) &:= \{0^\omega\} \cup \{p \in \mathbb{B} : (\exists i \in \mathbb{N})\, p = 0^i 1 0^\omega\}, \\
\mathrm{LLPO}_n[\langle p^{(0)}, \ldots, p^{(n-1)} \rangle_n] &:= \{i \in [n] : p^{(i)} = 0^\omega\},
\end{aligned}$$

for all $p^{(0)}, \ldots, p^{(n-1)} \in \mathbb{B}$ such that $\langle p^{(0)}, \ldots, p^{(n-1)} \rangle_n \in \mathrm{dom}(\mathrm{LLPO}_n)$.




$\mathcal{F}_2(F_{\mathrm{LLPO}_3})$ $\mathcal{F}_2(F_{\mathrm{LLPO}_2})$

# Trees Describing the Discontinuities of Functions

## Example

Let $Y := \mathbb{N} \cup \{\infty\}$, and let $\preceq$ be the equality relation on $Y$. Let $\min_d : \mathbb{B} \to Y$ be defined by

$$\min_d(p) := \begin{cases} \min\{p(i) - 1 \ : \ i \in \mathbb{N}\} & \text{if } p \neq 0^\omega, \\ \infty & \text{if } p = 0^\omega, \end{cases}$$

for $p \in \mathbb{B}$.

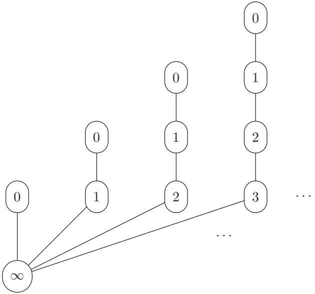



Figure: A representant of $\mathcal{F}_{\omega+1}(\min_d)$.

# Discontinuity Forests and Admissible Representations

## Proposition

*Let $X$ be a second countable $T_0$-space with an admissible representation $\rho :\subseteq \mathbb{B} \to X$. Let $(Y, \preceq)$ be a better-quasi-ordered set. Then for all countable ordinals $\alpha$ and all functions $f :\subseteq X \to Y$ with countable range*

$$\mathcal{F}_\alpha(f \circ \rho) = \mathcal{F}_\alpha(f).$$

# Topological Degree Structures for Functions with Values in a Better-quasi-order

### Theorem

*Let $X'$ be a separable metric space, let $(Y, \preceq)$, $(Y', \preceq')$ be quasi-ordered sets, and let $f :\subseteq \mathbb{B} \to Y$ and $f' :\subseteq X' \to Y$ be functions with countable range and $\alpha$ be a countable ordinal such that $\mathcal{F}_{\alpha+1}(f)$ exists and $\mathcal{F}_{\alpha+1}(f) = \mathcal{F}_{\alpha}(f)$ and $\mathcal{F}_{\alpha}(f')$ exists.*
*Then, for $i = 0, 1, 2$, TFAE:*

1. $f \preceq_i f'$.
2. $\mathcal{F}_{\alpha}(f) \preceq_i \mathcal{F}_{\alpha}(f')$,

# Topological Degree Structures for Functions with Values in a Better-quasi-order

## Theorem (H., Selivanov, Kihara/Montalban)

*Let $(Y, \preceq)$ be a better-quasi-ordered set. Then, for $i = 0, 1, 2$, the following posets are isomorphic.*

1. *the $\preceq_i$-degree structure of functions $f : \mathbb{B} \to Y$ that are $\Sigma^0_2$-measurable with respect to the reverse Alexandroff topology on $Y$ and that have countable range,*
2. *the $\preceq_i$-degree structure of functions $f : \mathbb{B} \to Y$ that are $\Delta^0_2$-measurable with respect to the discrete topology on $Y$,*
3. $(\mathrm{Forestclasses}(Y) \setminus \{\emptyset\}, \preceq_i)$.

# Wadge Degrees of Multivalued Functions with Finite Range

## Theorem

*Let $X, X'$ be second-countable $T_0$-spaces.*
*Let $Y$ be a finite set.*
*Let $R :\subseteq X \rightrightarrows Y$ be a multivalued function such that there exists a countable ordinal $\alpha$ such that $\mathcal{F}_\alpha(F_R) = \mathcal{F}_{\alpha+1}(F_R)$.*
*Let $R' :\subseteq X' \rightrightarrows Y'$ be a multivalued function.*
*Choose $F = (T, \lambda) \in \mathcal{F}_\alpha(F_R)$ and $F' = (T', \lambda') \in \mathcal{F}_\alpha(F_{R'})$.*
*Then TFAE:*

1. $R \leq_{\mathrm{Wa}} R'$,
2. $\mathcal{F}_\alpha(F_R) \supseteq_0 \mathcal{F}_\alpha(F_{R'})$.
3. *There exists a monotone function $h : T \to T'$ such that for all $t \in T$, $\lambda(t) \supseteq \lambda'(h(t))$.*

# Topological Strong Weihrauch Degrees of Multivalued Functions with Finite Range

**Theorem**

*Let $X, X'$ be second-countable $T_0$-spaces.*
*Let $Y, Y'$ be finite discrete spaces.*
*Let $R :\subseteq X \rightrightarrows Y$ be a multivalued function such that there exists a countable ordinal $\alpha$ such that $\mathcal{F}_\alpha(F_R) = \mathcal{F}_{\alpha+1}(F_R)$.*
*Let $R' :\subseteq X' \rightrightarrows Y'$ be a multivalued function .*
*Choose $F = (T, \lambda) \in \mathcal{F}_\alpha(F_R)$ and $F' = (T', \lambda') \in \mathcal{F}_\alpha(F_{R'})$.*
*Then TFAE:*

1. $R \leq_{\mathrm{sW}} R'$,
2. $\mathcal{F}_\alpha(F_R) \supseteq_1 \mathcal{F}_\alpha(F_{R'})$,
3. *There exists a monotone function $h : T \to T'$ such that for all nonempty $M \subseteq T$, if $\bigcap_{m \in M} \lambda(m) = \emptyset$ then $\bigcap_{m \in M} \lambda'(h(m)) = \emptyset$.*

# Topological Weihrauch Degrees of Multivalued Functions with Finite Range

**Theorem**

*Let $X, X'$ be second-countable $T_0$-spaces.*
*Let $Y, Y'$ be finite discrete spaces.*
*Let $R :\subseteq X \rightrightarrows Y$ be a multivalued function such that there exists a countable ordinal $\alpha$ such that $\mathcal{F}_\alpha(F_R) = \mathcal{F}_{\alpha+1}(F_R)$.*
*Let $R' :\subseteq X' \rightrightarrows Y'$ be a multivalued function .*
*Choose $F = (T, \lambda) \in \mathcal{F}_\alpha(F_R)$ and $F' = (T', \lambda') \in \mathcal{F}_\alpha(F_{R'})$.*
*Then TFAE:*

1. $R \leq_{\mathrm{W}} R'$,
2. $\mathcal{F}_\alpha(F_R) \supseteq_2 \mathcal{F}_\alpha(F_{R'})$,
3. *There exists a monotone function $h : T \to T'$ such that for all nonempty $M \subseteq T$ that have a smallest element, if $\bigcap_{m \in M} \lambda(m) = \emptyset$ then $\bigcap_{m \in M} \lambda'(h(m)) = \emptyset$.*

# Final Comments

- We have described the topological Wadge/strong Weihrauch/Weihrauch degrees for all $\Delta^0_2$-functions $f : \mathbb{B} \to Y$ ($Y$ a better-quasi-ordered set) and corresponding relations with finite discrete range in a combinatorial way.

  To be done: Extend this to topologically more complicated functions $f : \mathbb{B} \to Y$ (following Selivanov and Kihara/Montalban)!
- So far only for functions/relations with range in a better-quasi-ordered set or in a finite space (a smaller initial segment also for functions with countably infinite discrete range).

  To be done: Extend this to functions/relations with continuous range!
- Is such an analysis of the discontinuities of computation problems useful for the practice of computing?