A Constructive Model of Uniform Continuity

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TLCA 2013, 26-28 June 2013, Eindhoven, The Netherlands

Kleene–Kreisel continuous functionals

The model of Kleene–Kreisel continuous functionals validates the uniform continuity axiom (UC):

$$\forall f\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}.\ \exists n \in \mathbb{N}.\ \forall \alpha, \beta \in \mathbf{2}^{\mathbb{N}}.\ (\alpha =_n \beta \implies f(\alpha) = f(\beta)).$$

But the treatment of the model is non-constructive.

Making it constructive

We introduce a classically equivalent model, $\mathbb{C}$-spaces,

- in which the uniform continuity axiom holds,
- but without assuming any constructively contentious axiom in the meta-theory used to define the model.

We work with an intensional type theory with a universe, Σ-types, Π-types, identity types and standard base types.

We have formalized our development and proofs in Agda.

In this talk, however, I will use informal, rigorous mathematical language.

Precursors of our work

- Spanier's paper quasi-topological spaces (1961).
- Johnstone's paper On a topological topos (1979).
- Fourman's papers
  Notions of choice sequence (1982) and
  Continuous truth (1984).
- van der Hoeven and Moerdijk's paper
  Sheaf models for choice sequences (1984).
- Bauer and Simpson's unpublished work
  Continuity begets continuity (2006).
- Coquand and Jaber's papers
  A note on forcing and type theory (2010) and
  A computational interpretation of forcing in type theory (2012).

Presheaves and natural transformations

The Kleene–Kreisel functionals will arise as a full subcategory of a sheaf topos.

The underlying category of the site is

> the monoid C of uniformly continuous maps $\mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$

A presheaf can be described as a set P equipped with an action

$$((p,t) \mapsto p \cdot t) \colon P \times \mathrm{C} \to P$$

satisfying

$$p \cdot \mathrm{id} = p, \quad p \cdot (t \circ u) = (p \cdot t) \cdot u.$$

A natural transformation of presheaves $(P, \cdot)$ and $(Q, \cdot)$ is a function $\phi \colon P \to Q$ that preserves the action, i.e.

$$\phi(p \cdot t) = (\phi\ p) \cdot t.$$

The coverage

Let 2^n denote the set of binary strings of length n.

For $s \in 2^n$, let $\mathrm{cons}_s \colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ denote the concatenation map

$$\mathrm{cons}_s(\alpha) = s\alpha.$$

For each natural number n we have the covering family $\langle \mathrm{cons}_s \rangle_{s \in 2^n}$.

The coverage axiom, saying that for every $t \in \mathrm{C}$,

$$\boxed{\forall m \in \mathbb{N}.\ \exists n \in \mathbb{N}.\ \forall s \in 2^n.\ \exists t' \in \mathrm{C}.\ \exists s' \in 2^m.\ t \circ \mathrm{cons}_s = \mathrm{cons}_{s'} \circ t'}$$

holds because it is equivalent to the uniform continuity of every $t \in \mathrm{C}$.

Sheaves

A presheaf $(P, \cdot)$ is a sheaf if and only if for any n and any family $\langle p_s \in P \rangle_{s \in 2^n}$, there is a unique $p \in P$ such that for all $s \in 2^n$

$$p \cdot \mathrm{cons}_s = p_s.$$

It suffices to consider the case $n = 1$.

A presheaf $(P, \cdot)$ is a sheaf if and only if for any two $p_0, p_1 \in P$, there is a unique $p \in P$ with

$$p \cdot \mathrm{cons}_0 = p_0, \quad p \cdot \mathrm{cons}_1 = p_1.$$

Concrete sheaves

A sheaf $(P, \cdot)$ is concrete if the action $(\cdot)$ is function composition.

Then P must be a set of functions $\mathbf{2}^{\mathbb{N}} \to X$ for a suitable set X.

Concrete sheaves can be regarded as spaces, and their natural transformations as continuous maps.

C-spaces and continuous maps

Def. A C-topology on a set X is a collection P of probes $\mathbf{2}^{\mathbb{N}} \to X$ subject to the following conditions:

1. All constant maps are in P.
2. If $t\colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ is uniformly continuous and $p \in P$, then $p \circ t \in P$. (Presheaf condition)
3. For any two maps $p_0, p_1 \in P$, the unique map $p\colon \mathbf{2}^{\mathbb{N}} \to X$ defined by $p(i * \alpha) = p_i(\alpha)$ is in P. (Sheaf condition)

A C-space is a set X equipped with C-topology.

A function $f\colon X \to Y$ of C-spaces is continuous if $f \circ p \in P_Y$ whenever $p \in P_X$. (Naturality condition)

Examples of $\mathbb{C}$-spaces

All continuous maps from $\mathbf{2}^{\mathbb{N}}$ (with the usual topology) to any topological space X form a $\mathbb{C}$-topology on X:

- Any constant map $\mathbf{2}^{\mathbb{N}} \to X$ is continuous.
- The composite $\mathbf{2}^{\mathbb{N}} \xrightarrow{t} \mathbf{2}^{\mathbb{N}} \xrightarrow{p} X$ of two continuous maps is continuous.
- The sheaf condition is satisfied because $\mathbf{2}^{\mathbb{N}}$ is compact Hausdorff and each covering family $\langle \mathrm{cons}_s \rangle_{s \in 2^n}$ is finite and jointly surjective.

Any continuous map of topological spaces is continuous w.r.t. the above $\mathbb{C}$-topology, as composition preserves continuity.

$\mathbb{C}$-spaces form a cartesian closed category

The constructions are the same as in the category of sets, with suitable $\mathbb{C}$-topologies. For example,

1. to get products, we $\mathbb{C}$-topologize cartesian products,
2. to get exponentials, we $\mathbb{C}$-topologize the sets of continuous maps.

These constructions are different from those needed to get cartesian closedness of sheaves, but isomorphic.

They are simpler, which is good for our formalization purposes.

Discrete C-spaces

Def. A map $p\colon \mathbf{2}^{\mathbb{N}} \to X$ into a set X is called locally constant iff
$\exists n \in \mathbb{N}.\ \forall \alpha, \beta \in \mathbf{2}^{\mathbb{N}}.\ (\alpha =_n \beta \implies p(\alpha) = p(\beta))$.

Lemma
The locally constant maps $\mathbf{2}^{\mathbb{N}} \to X$ form the finest C-topology on the set X.

Def. A C-space X is discrete if all functions $X \to Y$ into any C-space Y are continuous.

Lemma
A C-space is discrete iff its probes are precisely the locally constant functions.

Def. We thus refer to the collection of all locally constant maps $\mathbf{2}^{\mathbb{N}} \to X$ as the discrete C-topology on X.

Booleans and natural numbers object

The discrete $\mathbb{C}$-topology on $\mathbf{2}$ or $\mathbb{N}$ is the set of uniformly continuous maps.

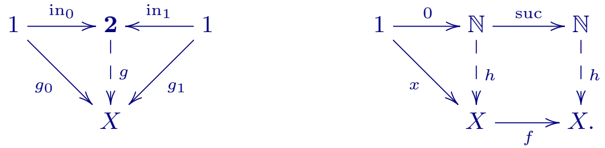
Theorem

In the category of $\mathbb{C}$-spaces:

1. The discrete space $\mathbf{2}$ is the coproduct of two copies of the terminal space.
2. The discrete space $\mathbb{N}$ is the natural numbers object.

Proof

The unique maps g and h in $\mathbf{Set}$ in the diagrams below are continuous by the discreteness of $\mathbf{2}$ and $\mathbb{N}$:



Model of Gödel's system T

At this stage we have all is needed to model T:

1. Cartesian closed structure (simply typed λ-calculus).
2. Natural numbers object (base type and primitive recursion principle).

(The local cartesian closed structure allows to further model dependent types.)

Yoneda Lemma

The monoid C itself is a C-topology on $\mathbf{2}^{\mathbb{N}}$.

The Yoneda embedding maps the monoid C to the C-space $(\mathbf{2}^{\mathbb{N}}, \mathrm{C})$. Moreover,

$$y(\star) = (\mathbf{2}^{\mathbb{N}}, \mathrm{C}) = \text{the exponential of the two discrete } \mathrm{C}\text{-spaces.}$$

The Yoneda Lemma says that a map $\mathbf{2}^{\mathbb{N}} \to X$ into a C-space X is a probe iff it is continuous in the sense of the category of C-spaces.

Uniform continuity of all Gödel's system T definable functions

Corollary

Any T-definable function $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ in $\mathbf{Set}$ is uniformly continuous.

Proof sketch

1. Define a logical relation between the $\mathbf{Set}$ and $\mathrm{C}\text{-}\mathbf{Space}$ interpretations of T.
2. A T-definable function $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ is related to a continuous map in $\mathrm{C}\text{-}\mathbf{Space}$.
3. By the Yoneda Lemma, this map is uniformly continuous.

The Fan functional

Lemma

The exponential $\mathbb{N}^{\mathbf{2}^{\mathbb{N}}}$ is a discrete C-space.

Theorem

There is a continuous functional $\mathrm{fan}\colon \mathbb{N}^{\mathbf{2}^{\mathbb{N}}} \to \mathbb{N}$ that calculates (minimal) moduli of uniform continuity.

Proof sketch

1. Because any $f \in \mathbb{N}^{\mathbf{2}^{\mathbb{N}}}$ is uniformly continuous, we can let $\mathrm{fan}(f)$ be the least witness of this fact.
2. Since $\mathbb{N}^{\mathbf{2}^{\mathbb{N}}}$ is discrete according to the above lemma, the functional fan is continuous.

Realizability of the uniform continuity axiom

We define a continuous realizability semantics of HA^ω inductively on formulas:

- Each HA^ω formula φ is associated with a T type $|\varphi|$.
- A continuous realizer of a formula $\Gamma \vdash \varphi$ is a pair

$$(\vec{q}, e) \in [\![\Gamma]\!] \times [\![|\varphi|]\!]$$

where $[\![-]\!]$ is the interpretation of T in C-$\mathbf{Space}$.

Theorem
The Fan functional realizes the uniform continuity axiom.

Kleene-Kreisel functionals via sequence convergence

One way of describing the Kleene–Kreisel spaces is to work with the cartesian closed category of Kuratowski limit spaces.

- A limit space is a set together with a designated set of convergent sequences, subject to suitable axioms.
- A function of limit spaces is called continuous if it preserves limits.
- To get the Kleene–Kreisel spaces, we start with the discrete natural numbers and iterate exponentials.

$\mathbb{C}$-spaces and limit spaces

Lemma

1. Limit spaces embed as a full subcategory of $\mathbb{C}$-spaces, with a left adjoint.
2. The embedding preserves the natural numbers object, products and exponentials.
3. On Kleene–Kreisel spaces, the embedding restricted to its image is an equivalence of categories.

$$\mathbf{Lim} \underset{G}{\overset{F}{\underset{\hookrightarrow}{\overset{\leftarrow}{\bot}}}} \mathbb{C}\text{-}\mathbf{Space}$$

Theorem

Kleene–Kreisel continuous functionals can be calculated within $\mathbb{C}$-$\mathbf{Space}$.

Summary

1. Calculating moduli of uniform continuity of T-definable functions $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$.
2. Validation of uniform continuity axiom in a weak constructive meta-theory.
3. Compatible with intuitionistic type theory and formalized in Agda.
4. Constructive treatment of Kleene–Kreisel functionals via a simple sheaf topos.

Spanier's quasi-topological spaces

Def. A quasi-topology on a set X assigns to each compact Hausdorff space K a set $P(K, X)$ of probes $p \colon K \to X$, such that:

1. All constant maps are in $P(K, X)$.
2. If $t \colon K' \to K$ is continuous and $p \in P(K, X)$, then $p \circ t \in P(K', X)$. (Presheaf condition)
3. If $\langle t_i \colon K_i \to K \rangle_{i \in I}$ is a finite, jointly surjective family and $p \colon K \to X$ is a function with $p \circ t_i \in P(K_i, X)$ for every $i \in I$, then $p \in P(K, X)$. (Sheaf condition)

A quasi-topological space is a set X equipped with quasi-topology.

A function $f \colon X \to Y$ of quasi-topological spaces is continuous if $f \circ p \in P(K, Y)$ whenever $p \in P(K, X)$. (Naturality condition)

Limit spaces

Def. A limit space consists of a set X together with a family of functions $x\colon \mathbb{N}_\infty \to X$, written as $(x_i) \to x_i nfty$ and called convergent sequences in X, satisfying the following conditions:

1. The constant sequence (x) converges to x.
2. If (x_i) converges to x_∞, then so does every subsequence of (x_i).
3. If (x_i) is a sequence such that every subsequence of (x_i) contains a subsequence converging to x_∞, then (x_i) converges to x_∞.

A function $f\colon X \to Y$ of limit spaces is said to be continuous if it preserves convergent sequences, i.e. $(x_i) \to x_\infty$ implies $(f(x_i)) \to f(x_i nfty)$.

C-spaces and limit spaces

Lemma

The continuous maps from $\mathbf{2}^{\mathbb{N}}$ (with the usual topology) to any topological space X form a C-topology on X. And any topologically continuous map is probe-continuous.

$\mathbb{N}_\infty$ with all continuous maps $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}_\infty$ is a C-space.

Lemma

The convergent sequences on a topological space form a limit structrure. And any topologically continuous map is limit-continuous.

$\mathbf{2}^{\mathbb{N}}$ with all convergent sequences is a limit space.

C-spaces and limit spaces (cont.)

- If X is a limit space, we can give a C-topology on it by saying that a map $\mathbf{2}^{\mathbb{N}} \to X$ is a probe on X iff it is limit-continuous.
- A map is probe-continuous if it is limit-continuous.
- $G \colon \mathbf{Lim} \to \mathrm{C}\text{-}\mathbf{Space}$

- If X is a C-space, we obtain its limit structure by saying that $(x_i) \to x_\infty$ in X iff the induced function $x \colon \mathbb{N}_\infty \to X$ is probe-continuous.
- A map is limit-continuous if it is probe-continuous.
- $F \colon \mathrm{C}\text{-}\mathbf{Space} \to \mathbf{Lim}$.

C-spaces and limit spaces (cont.)

$$\mathbf{Lim} \underset{G}{\overset{F}{\underset{\hookrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{C}\text{-}\mathbf{Space}$$

Lemma

Limit spaces form an exponential ideal of C-spaces and are closed under finite products.

Lemma

If X is a discrete C-space, then $G(F(X)) = X$.

(Proved using classical logic.)