Sheaf models of type theory in type theory

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Motivation

- Sheaf toposes (on different sites) for studying non-classical principles, *e.g.* the work of Johnstone (1979), Fourman (1982), van der Hoeven and Moerdijk (1984), Esacardó and Xu (2015), and Kawai (M4C).
- Sheaf models of nonstandard arithmetic, *e.g.* the work of Moerdijk (1995), Palmgren (1997), Hadzihasanovic and van den Berg (2014).
- (Pre)sheaves as models of Martin-Löf type theory (MLTT), *e.g.* sheaf models of MLTT + nonclassical principles (or nonstandard axioms?), the cubical set model of univalence (Bezem, Coquand and Huber 2014), the presheaf model of guarded cubical type theory (Spitters *et. al*. 2016).
- Constructive models are expected to be formalisable within MLTT.
- formalising (pre)sheaf models provides formal verifications of the above.
- Development in intensional MLTT gives runnable programs – computation!
- Compatibility of principles via their models.

Exmaple (Escardó & Xu 2015)

A sheaf topos $\mathbf{Shv}(\mathrm{C}, \mathscr{J})$, similar to Johnstone's topological topos $\mathscr{E}$

Concrete sheaves are C-spaces (similarly, those in $\mathscr{E}$ are limit spaces)

A C-space $X \equiv (|X|, \mathrm{Prb}(X))$ where $\mathrm{Prb}(X)$ a collection of maps $\mathbf{2}^{\mathbb{N}} \to |X|$, called probes, satisfying certain conditions preserved by the constructions of $\mathbf{2}$, $\mathbb{N}$, $\to$, $\times$, Π, Σ.

Yoneda Lemma: $$\mathrm{C}\text{-}\mathbf{Space}(\mathbf{2}^{\mathbb{N}}, X) \;\cong\; \mathrm{Prb}(X)$$

The probes on $\mathbb{N}$ are precisely the uniformly continuous $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$.

Hence, every continuous function $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ in $\mathrm{C}\text{-}\mathbf{Space}$ is uniformly continuous.

Exmaple (Escardó & Xu 2015) (cont.)

A Gödel's T term $f : (\mathsf{N} \to 2) \to \mathsf{N}$ (or a term in MLTT)

$\downarrow$ $[\![-]\!]$

A continuous map $[\![f]\!] : \mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ in C-$\mathbf{Space}$

$\downarrow$ Yoneda Lemma

$[\![f]\!]$ is uniformly continuous (as Σ)

$\downarrow$ $\mathrm{pr}_1, \mathrm{pr}_2$

The least modulus of uniform continuity of f

An Agda program

The aims of this ongoing study

- To investigate the feasibility of developing (pre)sheaf models of MLTT with Σ-types, Π-types, identity types and universes within intensional type theory.
- To identify necessary extensions for the development, *e.g.* function extensionality, uniqueness of identity proofs, univalence axiom.

Category with families (CwF) (Dybjer 1995)

A base category $\mathbf{C}$ of contexts and substitutions with a terminal object.

A functor $T : \mathbf{C}^{\mathrm{op}} \to \mathbf{Fam}$ for types, terms and their substitutions, mapping

- a context $\Gamma \in \mathbf{C}$ to a family of sets $\{\mathrm{Term}(\Gamma, \mathrm{A})\}_{\mathrm{A} \in \mathrm{Type}(\Gamma)}$,
- a substitution $\sigma : \Delta \to \Gamma$ to a $\mathbf{Fam}$-morphism consisting of
  - a type substitution map $(A \mapsto A[\sigma]) : \mathrm{Type}(\Gamma) \to \mathrm{Type}(\Delta)$
  - a family of term substitution maps $(u \mapsto u[\sigma]) : \mathrm{Term}(\Gamma, A) \to \mathrm{Term}(\Delta, A[\sigma])$ for each $A \in \mathrm{Type}(\Gamma)$.

An operation for context comprehension

- To each context $\Gamma \in \mathbf{C}$ and type $A \in \mathrm{Type}(\Gamma)$, it associates
  - a context $\Gamma.A \in \mathbf{C}$,
  - a substitution $\mathrm{p} : \Gamma.A \to \Gamma$,
  - a term $\mathrm{q} \in \mathrm{Term}(\Gamma.A, A[\mathrm{p}])$
- For any substitution $\sigma : \Delta \to \Gamma$ and term $u \in \mathrm{Term}(\Delta, A[\sigma])$, there is a unique substitution $(\sigma, u) : \Delta \to \Gamma.A$ satisfying

$$\mathrm{p} \circ (\sigma, u) = \sigma \qquad \mathrm{q}[(\sigma, u)] = u.$$

Example: the CwF of sets

$$
\begin{array}{rrcl}
 & \mathbf{C} & := & \mathbf{Set} \\
\mathrm{Type}(\Gamma) \ni & A & := & \{A_\gamma\}_{\gamma\in\Gamma} \\
\mathrm{Type}(\Delta) \ni & A[\sigma] & := & \{A_{\sigma(\delta)}\}_{\delta\in\Delta} \\
\mathrm{Term}(\Gamma, A) \ni & u & := & u : \prod_{\gamma\in\Gamma} A_\gamma \\
\mathrm{Term}(\Delta, A[\sigma]) \ni & u[\sigma] & := & u\circ\sigma : \prod_{\delta\in\Delta} A_{\sigma(\delta)} \\
\mathbf{Set} \ni & \Gamma.A & := & \sum_{\gamma\in\Gamma} A_\gamma \\
 & \mathrm{p} & := & \mathrm{pr}_1 : \sum_{\gamma\in\Gamma} A_\gamma \to \Gamma \\
\mathrm{Term}(\Gamma.A, A[\mathrm{p}]) \ni & \mathrm{q} & := & \mathrm{pr}_2 : \prod_{w\in\sum_{\gamma\in\Gamma} A_\gamma} A_{\mathrm{pr}_1(w)} \\
 & (\sigma, u) & := & \lambda\delta.(\sigma(\delta), u(\delta)) : \Delta \to \sum_{\gamma\in\Gamma} A_\gamma
\end{array}
$$

Presheaf models – CwFs of presheaves (Coquand's note)

A presheaf on a category $\mathbf{C}$ is a functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$.

For simplicity, here we consider only the presheaves on a monoid $(\mathrm{M}, 1, \circ)$.

A presheaf on M can be represented as a set Γ equipped with an action

$$((\gamma, t) \mapsto \gamma \cdot t) : \Gamma \to \mathrm{M} \to \Gamma$$

such that, for all $\gamma \in \Gamma$ and $t, r \in \mathrm{M}$

$$\gamma \cdot 1 = \gamma \qquad (\gamma \cdot t) \cdot r = \gamma \cdot (t \circ r).$$

A natural transformation of presheaves is a map $\sigma : \Delta \to \Gamma$ such that

$$\sigma(\delta) \cdot t = \sigma(\delta \cdot t)$$

for all $\delta \in \Delta$ and $t \in \mathrm{M}$.

Presheaf models (cont.)

Given a presheaf Γ, a type $A \in \mathrm{Type}(\Gamma)$ is a Γ-indexed family of sets $\{A_\gamma\}_{\gamma\in\Gamma}$ equipped with a restriction map

$$((a,t) \mapsto a * t) : A_\gamma \to \prod_{t \in \mathrm{M}} A_{\gamma\cdot t}$$

for each $\gamma \in \Gamma$, such that, for any $\gamma \in \Gamma, a \in A_\gamma$ and $t, r \in \mathrm{M}$

$$a * 1 = a \qquad (a * t) * r = a * (t \circ r).$$

A term $u \in \mathrm{Term}(\Gamma, A)$ is a dependent function

$$u : \prod_{\gamma\in\Gamma} A_\gamma$$

such that, for all $\gamma \in \Gamma$ and $t \in \mathrm{M}$

$$u(\gamma) * t = u(\gamma \cdot t).$$

Some type-theoretic preliminaries

We attempt to develop the CwF of presheaves in intensional type theory, using identity types to formulate the equations of the construction.

We write $a = b$ to denote the intensional identity type $\mathrm{Id}_A(a, b)$, and $\mathcal{U}$ to denote the universe of all (small) types.

For the underlying monoid, we assume the followings are given:

- a type $\mathrm{M} : \mathcal{U}$,
- an element $1 : \mathrm{M}$,
- an operation $\_ \circ \_ : \mathrm{M} \to \mathrm{M} \to \mathrm{M}$,
- a proof $\mathrm{id}_\mathrm{M} : \Pi(t \colon \mathrm{M}).\ t \circ 1 = t$,
- a proof $\mathrm{id}'_\mathrm{M} : \Pi(t \colon \mathrm{M}).\ 1 \circ t = t$, and
- a proof $\mathrm{assoc}_\mathrm{M} : \Pi(t, r, s \colon \mathrm{M}).\ (t \circ r) \circ s = t \circ (r \circ s)$.

A native formulation of presheaves and natural transformations

The type of presheaves

$$\mathrm{PSh} \;:=\; \Sigma(\Gamma : \mathcal{U}).\ \mathrm{isPSh}(\Gamma)$$

where

$$\begin{aligned}\mathrm{isPSh}(\Gamma) \;:=\; \Sigma(\_ \cdot \_ : \Gamma \to \mathrm{M} \to \Gamma).&\\ &(\Pi(\gamma : \Gamma).\ \gamma \cdot 1 = \gamma)\\ &\times (\Pi(\gamma : \Gamma)(t, r : \mathrm{M}).\ (\gamma \cdot t) \cdot r = \gamma \cdot (t \circ r))\end{aligned}$$

The type of natural transformations of $\Delta, \Gamma : \mathrm{PSh}$

$$\mathrm{Nat}(\Delta, \Gamma) \;:=\; \Sigma(\sigma : \Delta \to \Gamma).\ \Pi(\delta : \Delta)(t : \mathrm{M}).\ (\sigma\delta) \cdot t = \sigma(\delta \cdot t)$$

A problematic formulation of types

Given $\Gamma : \mathrm{PSh}$,

$$\mathrm{Type}(\Gamma) \;:=\; \Sigma(A : \Gamma \to \mathcal{U}).\; \mathrm{isType}(A)$$

where

$$\begin{aligned}\mathrm{isType}(A) \;:=\; \Sigma(\_ * \_ : \Pi\{\gamma : \Gamma\}.\, A_\gamma \to \Pi(t : \mathrm{M}).A_{\gamma\cdot t}).&\\ (\Pi(\gamma : \Gamma)(a : A_\gamma).\; a * 1 = a)&\\ \times(\Pi(\gamma : \Gamma)(a : A_\gamma)(t, r : \mathrm{M}).\; (a * t) * r = \gamma * (t \circ r))&\end{aligned}$$

This does not type-check!

For instance, we can't form $a * 1 = a$ because $a * 1 : A_{\gamma\cdot 1}$ and $a : A_\gamma$ have different types.

More type-theoretic preliminaries

To make it type-check, we transport the element in one side of the equation, using

$$\mathrm{transport}(p, \text{-}) : P(a) \to P(b)$$

where $P : A \to \mathcal{U}$ and $p : a = b$.

Given $\Gamma : \mathrm{PSh}$, we have two witnesses

$$\mathrm{id}_\Gamma : \Pi(\gamma\!:\!\Gamma).\ \gamma \cdot 1 = 1 \qquad \mathrm{assoc}_\Gamma : \Pi(\gamma\!:\!\Gamma)(t, r\!:\mathrm{M}).\ (\gamma \cdot t) \cdot r = \gamma \cdot (t \circ r)$$

Then the two equations in isType can be formulated as

$$a * 1 =^{\mathrm{id}_\Gamma(\gamma)} a \qquad (a * t) * r =^{\mathrm{assoc}_\Gamma(\gamma, t, r)} a * (t \circ r)$$

where we write $x =^p y$ to denote $\mathrm{transport}(p, x) = y$.

A less problematic formulation

Given $\Gamma : \mathrm{PSh}$,

$$\mathrm{Type}(\Gamma) \ := \ \Sigma(A : \Gamma \to \mathcal{U}).\ \mathrm{isType}(A)$$

where

$$\begin{aligned}\mathrm{isType}(A) \ := \ & \Sigma(\_ * \_ : \Pi\{\gamma{:}\Gamma\}.\ A_\gamma \to \Pi(t{:}\,\mathrm{M}).A_{\gamma\cdot t}).\\ & \quad (\Pi(\gamma{:}\Gamma)(a{:}A_\gamma).\ a * 1 =^{\mathrm{id}_\Gamma} a)\\ & \quad \times (\Pi(\gamma{:}\Gamma)(a{:}A_\gamma)(t, r{:}\,\mathrm{M}).\ (a * t) * r =^{\mathrm{assoc}_\Gamma(\gamma,t,r)} \gamma * (t \circ r))\end{aligned}$$

Given $A : \mathrm{Type}(\Gamma)$, we get two witnesses

$$\mathrm{id}_A : \Pi(\gamma{:}\Gamma)(a{:}A_\gamma).\ a * 1 =^{\mathrm{id}_\Gamma} a$$

$$\mathrm{assoc}_A : \Pi(\gamma{:}\Gamma)(a{:}A_\gamma)(t, r{:}\,\mathrm{M}).\ (a * t) * r =^{\mathrm{assoc}_\Gamma(\gamma,t,r)} \gamma * (t \circ r)$$

A problem in type substitutions

Given $A : \mathrm{Type}(\Gamma)$ and $\sigma : \mathrm{Nat}(\Delta, \Gamma)$, the substituted type $A[\sigma] : \mathrm{Type}(\Delta)$ is given by the type family $A[\sigma] : \Delta \to \mathcal{U}$ defined by, for $\delta : \Delta$,

$$A[\sigma]_\delta := A_{\sigma(\delta)}.$$

Given $a : A[\sigma]_\delta$ and $t : \mathrm{M}$, we can't simply define

$$a *_{A[\sigma]} t : A_{\sigma(\delta \cdot t)}$$

to be $a *_A t : A_{(\sigma\delta)\cdot t}$.
But we can transport it

$$a *_{A[\sigma]} t \;:=\; \mathrm{transport}(\mathrm{nat}_\sigma(\delta, t), a *_A t)$$

where $\mathrm{nat}_\sigma : \Pi(\delta{:}\Delta)(t{:}\,\mathrm{M}).\ (\sigma\delta) \cdot t = \sigma(\delta \cdot t)$.

A problem in type substitutions (cont.)

It remains to construct $\mathrm{id}_{A[\sigma]}$ and $\mathrm{assoc}_{A[\sigma]}$, which is impossible without further adjustments.

For instance, the type of $\mathrm{id}_{A[\sigma]}(\delta, a)$ is expanded to

$$\mathrm{transport}(\mathrm{nat}_{\sigma}(\delta, 1) \bullet \mathrm{ap}(\sigma, \mathrm{id}_{\Delta}(\delta)), a *_A 1) = a$$

where $p \bullet q : x = z$ is the concatenation of $p : x = y$ and $q : y = z$, and $\mathrm{ap}(f, p) : fx = fy$ applies the map f to $p : x = y$.

We only have

$$\mathrm{id}_A(\sigma\delta, a) : \mathrm{transport}(\mathrm{id}_{\Gamma}(\sigma\delta), a *_A 1) = a$$

but we cannot prove

$$\mathrm{nat}_{\sigma}(\delta, 1) \bullet \mathrm{ap}(\sigma, \mathrm{id}_{\Delta}(\delta)) = \mathrm{id}_{\Gamma}(\sigma\delta).$$

First attempt – restricting natural transformations

To construct $\mathrm{id}_{A[\sigma]}$, we need

$$E_{\mathrm{id}}(\mathrm{nat}_\sigma) := \Pi(\delta : \Delta).\ \ \mathrm{nat}_\sigma(\delta, 1) \bullet \mathrm{ap}(\sigma, \mathrm{id}_\Delta(\delta)) = \mathrm{id}_\Gamma(\sigma\delta)$$

(and, similarly, an $E_{\mathrm{assoc}}(\mathrm{nat}_\sigma)$ for constructing $\mathrm{assoc}_{A[\sigma]}$).

We attempt to refine natural transformations by

$$\begin{aligned}\mathrm{Nat}(\Delta, \Gamma) \ := \ & \Sigma(\sigma : \Delta \to \Gamma). \\ & \qquad \Sigma(\mathrm{nat}_\sigma : \Pi(\delta{:}\Delta)(t{:}\,\mathrm{M}).\ (\sigma\delta) \cdot t = \sigma(\delta \cdot t)). \\ & \qquad\quad E_{\mathrm{id}}(\mathrm{nat}_\sigma) \times E_{\mathrm{assoc}}(\mathrm{nat}_\sigma)\end{aligned}$$

But it introduces new problems: we can't prove

$$\sigma \circ 1 = \sigma \qquad (\sigma \circ \tau) \circ \nu = \sigma \circ (\tau \circ \nu)$$

because there is no reason why one can have *e.g.* $E_{\mathrm{id}}(\mathrm{nat}_{\sigma\circ 1}) = E_{\mathrm{id}}(\mathrm{nat}_\sigma)$.

Second attempt – restricting presheaves

A type $A : \mathcal{U}$ is a set if

$$\mathrm{isSet}(A) := \Pi(x, y : A)(p, q : x = y).\ p = q.$$

We refine the formulations by

$$\begin{aligned} \mathrm{PSh} &:= \Sigma(\Gamma : \mathcal{U}).\ \mathrm{isSet}(\Gamma) \times \mathrm{isPSh}(\Gamma) \\ \mathrm{Type}(\Gamma) &:= \Sigma(A : \Gamma \to \mathcal{U}).\ (\Pi(\gamma : \Gamma).\, \mathrm{isSet}(A_\gamma)) \times \mathrm{isType}(A) \end{aligned}$$

We need function extensionality (available in Cubical TT) to show that the underlying type family of a Π-type in the CwF of presheaves is set-valued.

But we can't construct universes of presheaves, because $\mathcal{U}$ is not a set.

We can also work with UIP or Streicher's K-axiom (available in Agda).

Third attempt – using setoids

Use equivalence relations to formulate the equations.

Equivalence relations has to be proposition-valued. Otherwise, again we cannot construct $\mathrm{id}_{A[\sigma]}$ and $\mathrm{assoc}_{A[\sigma]}$.

But we still cannot construct universes of presheaves, because the equivalence relation on $\mathcal{U}$ is isomorphism which is not proposition-valued.

Universes in presheaf models

The monoid M is also a presheaf. We define the universe $\mathrm{U} \in \mathrm{Type}(\Gamma)$ by

$$\mathrm{U}_\gamma \ := \ \mathrm{Type}(\mathrm{M})$$

for all $\gamma \in \Gamma$. Given $T \in \mathrm{U}_\gamma$ and $t \in \mathrm{M}$, we define

$$(T * t)r \ := \ T_{t \circ r}.$$

for any $r \in \mathrm{M}$.

$\mathrm{Type}(\mathrm{M})$ consists of families $T : \mathrm{M} \to \mathcal{U}$ satisfying certain condition (formulated as a Σ-type). We cannot prove that $\mathrm{Type}(\mathrm{M})$ is a set, because we cannot prove $\mathcal{U}$ (or the subuniverse of sets) to be a set, unless we assume K-axiom.

Truncating $\mathrm{Type}(\mathrm{M})$ to a set $\| \mathrm{Type}(\mathrm{M}) \|_0$ does not work; otherwise, the decoding operation $\mathrm{EL} : \mathrm{Term}(\Gamma, \mathrm{U}) \to \mathrm{Type}(\Gamma)$ will be able to turn sets back to types.

Universers in sheaf models

A coverage $\mathcal{J}$ on a monoid M is a collection of subsets of M, called the covering families, satisfying the coverage axiom:

for any $I \in \mathcal{J}$ and $t \in \mathrm{M}$, there exists a $J \in \mathcal{J}$ such that for each $j \in J$ there are $i \in I$ and $r \in \mathrm{M}$ such that $t \circ j = i \circ r$.

A presheaf Γ is a sheaf on $(\mathrm{M}, \mathcal{J})$ if it satisfies the sheaf condition:

for any $I \in \mathcal{J}$ and any compatible family of elements $\{\gamma_i \in \Gamma \mid i \in I\}$ there exists a unique amalgamation $\gamma \in \Gamma$ such that $\gamma \cdot i = \gamma_i$ for all $i \in I$.

We need to add a (dependent) sheaf condition to the definition of types in the CwF of sheaves.

When verifying the sheaf condition of the universe, we can only show that the amalgamation of a family of elements in $\mathrm{Type}(\mathrm{M})$ is unique up to (pointwise) isomorphism.

So we need (a weaker form of) the univalence axiom (UA)?

But UA is inconsistent with K!

Summaries

- Developing (pre)sheaf models in intensional type theory (ITT) directly gives us correctness and computation.
- In ITT + FunExt, one can develop the CwF of (pre)sheaves without universes.
- In ITT + K, one can develop the CwF of presheaves with universes.
- Using setoids does not help too much.
- The construction of universes in the CwF of sheaves needs both K and UA which are inconsistent.