When is the Kleene–Kreisel Hierarchy Full?

Chuangjie Xu

joint work with Martín Escardó

University of Birmingham, UK

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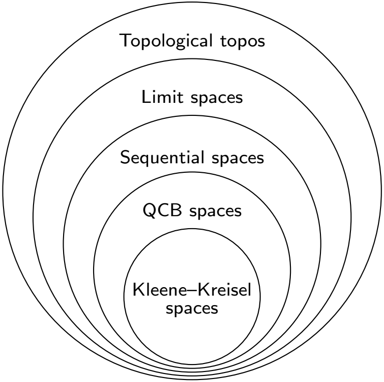
Kleene–Kreisel continuous functionals

First discussed as

- Kleene's countable functionals
- Kreisel's continuous functionals

Calculated within

- Compactly generated topological spaces
- Sequential topological spaces
- Simpson and Schröders QCB spaces
- Kuratowski limit spaces
- Filter spaces
- Scott's equilogical spaces
- Johnstone's topological topos
- ...



Validating the uniform continuity axiom

The model of Kleene–Kreisel continuous functionals validates the uniform continuity axiom (UC):

$$\forall f : \mathbf{2}^{\mathbb{N}} \to \mathbb{N}.\ \exists n \in \mathbb{N}.\ \forall \alpha, \beta \in \mathbf{2}^{\mathbb{N}}.\ (\alpha =_n \beta \implies f\alpha = f\beta).$$

But the treatment of the model is non-constructive.

Making it constructive

We introduce a classically equivalent model, $\mathbb{C}$-spaces,

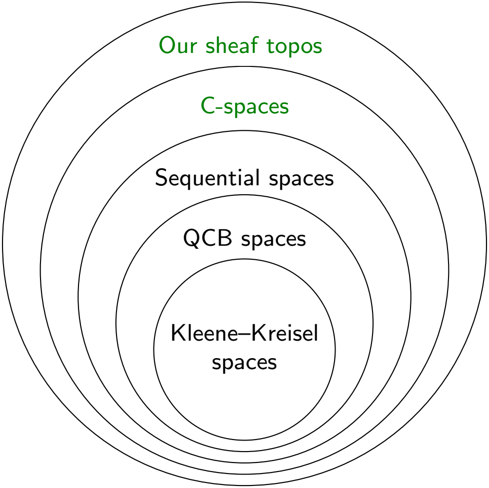
- in which the uniform continuity axiom holds,
- but without assuming any constructively contentious axiom in the meta-theory used to define the model.

We work with an intensional type theory with a universe, Σ-types, Π-types, identity types and standard base types.

We have formalized our development and proofs in Agda.

In this talk, however, I will use informal, rigorous mathematical language.

Our constructive manifestation



C-spaces and continuous maps

Def. A C-topology on a set X is a collection P of probes $\mathbf{2}^{\mathbb{N}} \to X$ subject to the following probe axioms:

1. All constant maps are in P.
2. If $t\colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ is uniformly continuous and $p \in P$, then $p \circ t \in P$. (Presheaf condition)
3. For any two maps $p_0, p_1 \in P$, the unique map $p\colon \mathbf{2}^{\mathbb{N}} \to X$ defined by $p(i * \alpha) = p_i(\alpha)$ is in P. (Sheaf condition)

A C-space is a set X equipped with C-topology.

A function $f\colon X \to Y$ of C-spaces is continuous if $f \circ p \in P_Y$ whenever $p \in P_X$. (Naturality condition)

We write C-**Space** for the category of C-spaces and continuous maps.

Examples of C-spaces

All continuous maps from $\mathbf{2}^{\mathbb{N}}$ (with the usual topology) to any topological space X form a C-topology on X:

- Any constant map $\mathbf{2}^{\mathbb{N}} \to X$ is continuous.
- The composite $\mathbf{2}^{\mathbb{N}} \xrightarrow{t} \mathbf{2}^{\mathbb{N}} \xrightarrow{p} X$ of two continuous maps is continuous.
- The sheaf condition is satisfied because $\mathbf{2}^{\mathbb{N}}$ is compact Hausdorff.

Any continuous map of topological spaces is continuous w.r.t. the above C-topology, as composition preserves continuity.

C-spaces form a (locally) cartesian closed category

The constructions are the same as in the category of sets, with suitable C-topologies. For example,

1. to get products in **C-Space**, we C-topologize cartesian products,
2. to get exponentials in **C-Space**, we C-topologize the sets of continuous maps,
3. to get products in **C-Space**$/X$, we C-topologize pullbacks,
4. to get exponentials in **C-Space**$/X$, we C-topologize the domains of exponentials in **Set**$/X$.

Moreover, **C-Space** has coproducts.

Discrete C-spaces

Def. A map $p\colon \mathbf{2}^{\mathbb{N}} \to X$ into a set X is called locally constant iff $\exists n \in \mathbb{N}.\ \forall \alpha, \beta \in \mathbf{2}^{\mathbb{N}}.\ (\alpha =_n \beta \implies p(\alpha) = p(\beta))$.

Lemma
The locally constant maps $\mathbf{2}^{\mathbb{N}} \to X$ form a C-topology which has the smallest amount of probes on X.

Def. A C-space X is discrete if all functions $X \to Y$ into any C-space Y are continuous.

Lemma
A C-space is discrete iff its probes are precisely the locally constant functions.

Def. We thus refer to the collection of all locally constant maps $\mathbf{2}^{\mathbb{N}} \to X$ as the discrete C-topology on X.

Booleans and natural numbers object

The discrete C-topology on $\mathbf{2}$ or $\mathbb{N}$ is the set of uniformly continuous maps.

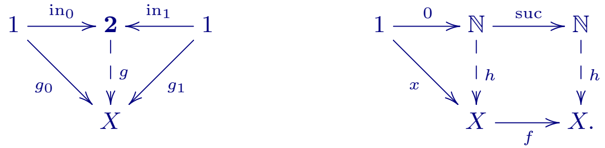
Theorem

In the category of C-spaces:

1. The discrete space $\mathbf{2}$ is the coproduct of two copies of the terminal space.
2. The discrete space $\mathbb{N}$ is the natural numbers object.

Proof

The unique maps g and h in $\mathbf{Set}$ in the diagrams below are continuous by the discreteness of $\mathbf{2}$ and $\mathbb{N}$:



Yoneda Lemma

The monoid C of uniformly continuous $\mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ is a C-topology on $\mathbf{2}^{\mathbb{N}}$.

The Yoneda embedding maps the monoid C to the C-space $(\mathbf{2}^{\mathbb{N}}, \mathrm{C})$. Moreover,

$$y(\star) = (\mathbf{2}^{\mathbb{N}}, \mathrm{C}) = \text{the exponential of the two discrete } \mathrm{C}\text{-spaces.}$$

The Yoneda Lemma says that a map $\mathbf{2}^{\mathbb{N}} \to X$ into a C-space X is a probe iff it is continuous in the sense of the category of C-spaces.

The Fan functional

Lemma
The exponential $\mathbb{N}^{\mathbf{2}^{\mathbb{N}}}$ is a discrete C-space.

Theorem
There is a continuous functional $\mathrm{fan}\colon \mathbb{N}^{\mathbf{2}^{\mathbb{N}}} \to \mathbb{N}$ that calculates (minimal) moduli of uniform continuity.

Proof sketch

1. Because any $f \in \mathbb{N}^{\mathbf{2}^{\mathbb{N}}}$ is uniformly continuous, we can let $\mathrm{fan}(f)$ be the least witness of this fact.
2. Since $\mathbb{N}^{\mathbf{2}^{\mathbb{N}}}$ is discrete according to the above lemma, the functional fan is continuous.

Modelling uniform continuity

C-spaces provide a model of system T and dependent types:

1. Cartesian closed structure — simply typed λ-calculus.
2. Locally cartesian closed structure — dependent types.
3. Natural numbers object — base type and primitive recursion principle.

Theorem
The uniform continuity axiom is validated by the fan functional.

$$\forall f\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}.\ \exists n \in \mathbb{N}.\ \forall \alpha, \beta \in \mathbf{2}^{\mathbb{N}}.\ (\alpha =_n \beta \implies f\alpha = f\beta).$$

Kleene-Kreisel functionals via sequence convergence

One way of describing the Kleene–Kreisel spaces is to work with the cartesian closed category of Kuratowski limit spaces.

- A limit space is a set together with a designated set of convergent sequences, subject to suitable axioms.
- A function of limit spaces is called continuous if it preserves limits.
- We write **Lim** for the category of limit spaces and continuous maps.

- To get the Kleene–Kreisel spaces, we start with the discrete natural numbers and iterate exponentials.

- Example: Any topological space with all (topologically) convergent sequences is a limit space.

C-spaces and limit spaces

- If A is a limit space, we can give a C-topology on it by saying that a map $\mathbf{2}^{\mathbb{N}} \to A$ is a probe on A iff it is limit-continuous.
- A map is probe-continuous if it is limit-continuous.
- $G\colon \mathbf{Lim} \to \mathrm{C}\text{-}\mathbf{Space}$

- If X is a C-space, we obtain its limit structure by saying that $(x_i) \to x_\infty$ in X iff the induced function $x\colon \mathbb{N}_\infty \to X$ is probe-continuous. (Uses non-constructive arguments.)
- A map is limit-continuous if it is probe-continuous.
- $F\colon \mathrm{C}\text{-}\mathbf{Space} \to \mathbf{Lim}$

- Both F and G keep the underlying set but change the structure, and are identity on morphisms.

Kleene–Kreisel functionals within C-spaces

Lemma

1. $G\colon \mathbf{Lim} \to \mathrm{C}\text{-}\mathbf{Space}$ is a full and faithful embedding.
2. $F\colon \mathrm{C}\text{-}\mathbf{Space} \to \mathbf{Lim}$ is left adjoint to G.

$$\mathbf{Lim} \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{C}\text{-}\mathbf{Space} \quad (F \dashv G)$$

   Thus $\mathbf{Lim}$ is a reflective subcategory of $\mathrm{C}\text{-}\mathbf{Space}$.

3. Moreover, the embedding G preserves the natural numbers object, and the cartesian closed structure (products and exponentials).

Theorem

Kleene–Kreisel continuous functionals can be calculated within $\mathrm{C}\text{-}\mathbf{Space}$.

If UC holds in the meta-theory

What does our model construction do?

Def.
The collection of all maps $\mathbf{2}^{\mathbb{N}} \to X$ form an indiscrete C-topology on X.
We say X is an indiscrete C-space.

Lemma

1. The category of indiscrete C-spaces is equivalent to $\mathbf{Set}$.
2. Indiscrete C-spaces form an exponential ideal of C-$\mathbf{Space}$.
3. If UC holds in $\mathbf{Set}$, then the discrete space $\mathbb{N}$ is also indiscrete.

The Kleene–Kreisel and full type hierarchies

Def.

Let $\mathbf{C}$ be a cartesian closed category with a natural numbers object.
The type hierarchy on $\mathbf{C}$ is the smallest full subcategory containing the natural numbers object and closed under exponentials.

The one on $\mathbf{Set}$ is called the full type hierarchy.
The one on C-$\mathbf{Space}$ is called the Kleene–Kreisel hierarchy.

Corollary
If UC holds in $\mathbf{Set}$, then
the Kleene–Kreisel hierarchy is equivalent to the full type hierarchy.

Theorem
The Kleene–Kreisel hierarchy and the full type hierarchy are equivalent
if and only if UC holds in $\mathbf{Set}$.

Summary

1. Validation of uniform continuity axiom in a weak constructive meta-theory.
2. Constructive manifestation of Kleene–Kreisel continuous functionals.
3. Equivalence of the Kleene–Kreisel hierarchy and the full type hierarchy when assuming UC.
4. Compatible with intuitionistic type theory and formalized in Agda.
5. Extraction of computational content from type-theoretic proofs which use UC.