Predicative Hierarchies

ABM – April 2019

Gerhard Jäger

University of Bern

Munich, April 2019

Outline of my talk

Predicativty – impredicativity

Russell and Poincaré (around 1901 – 1906)

- The vicious circle principle (VCP): A definition of an object S is *impredicative* if it refers to a totality to which S belongs.
- VPC is the essential source of inconsistencies.
- The structure of the natural numbers and the principle of induction on the natural numbers (for arbitrary properties) do not require foundational justification; further sets have to be introduced by purely predicative means.

Typical impredicative definitions

- $S \;=\; \{\, n \in \mathbb{N} : (\forall X \subseteq \mathbb{N})\varphi[X, n] \,\}$

  $$? : \quad m \in S \;\rightsquigarrow\; (\forall X \subseteq \mathbb{N})\varphi[X, m] \;\rightsquigarrow\; \varphi[S, m] \;\rightsquigarrow\; m \in S.$$

- Well-orderings

  Let $\prec$ be a (primitive recursive) linear ordering on $\mathbb{N}$ and X a subset of $\mathbb{N}$.

  $$Prog[\prec, X] \;:\Leftrightarrow\; (\forall m \in \mathbb{N})((\forall n \prec m)(n \in X) \to (m \in X)),$$

  $$Acc[\prec] \;:=\; \bigcap \{X \subseteq \mathbb{N} : Prog[\prec, X]\},$$

  $$WO[\prec] \;:\Leftrightarrow\; \mathbb{N} \subseteq Acc[\prec],$$

  ($WO[\prec] \;\Leftrightarrow\;$ every nonempty $X \subseteq \mathbb{N}$ has a $\prec$-least element).

Typical predicative definitions

Pick an arbitrary arithmetic formula $A[X, n]$ of second order arithmetic.

Arithmetic definitions. Consider the process

$$Pow(\mathbb{N}) \ni \; S \;\longmapsto\; \{n \in \mathbb{N} : \mathbb{N} \models A[S, n]\} \; \in Pow(\mathbb{N}).$$

Arithmetical hierarchies. Given a set $S \subseteq \mathbb{N}$ we write

$$m \in (S)_n \;:\Leftrightarrow\; \langle n, m \rangle \in S.$$

Now suppose that $\prec$ is a primitive recursive linear ordering such that 0 is its least element and

$$n \oplus 1 \text{ the successor of } n \text{ in } \prec.$$

We may also assume that the field of $\prec$ is $\mathbb{N}$.

Now suppose that

$$(S)_0 = \varnothing,$$

$$(S)_{n \oplus 1} = \{m \in \mathbb{N} : \mathbb{N} \models A[(S)_n, m]\},$$

$$(S)_\ell = \text{disjoint union of } (S)_n \text{ with } n \prec \ell \text{ if } \ell \text{ limit.}$$

Then we write $\mathcal{H}_A[\prec, S]$ and call S an A-hierarchy.

Question

For which linear orderings $\prec$ does this definition make sense?

First answer: well-orderings.

But is this enough if one wants to build up sets from below?

Further locally predicative hierarchies

Ramified analytic hierarchy

$$R_0 \;:=\; \emptyset, \qquad R_{\alpha+1} \;:=\; \mathit{Def}^{(2)}(R_\alpha), \qquad R_\lambda \;:=\; \bigcup_{\xi<\lambda} R_\xi \;\; (\lambda \text{ limit}).$$

Gödel's constructible hierarchy

$$L_0 \;:=\; \emptyset, \qquad L_{\alpha+1} \;:=\; \mathit{Def}(L_\alpha), \qquad L_\lambda \;:=\; \bigcup_{\xi<\lambda} L_\xi \;\; (\lambda \text{ limit}).$$

Every step $R_\alpha \mapsto R_{\alpha+1}$ and $L_\alpha \mapsto L_{\alpha+1}$ is justified from a predicative perspective.

Central question in connection with all these hierarchies:

How far are we allowed to iterate?

A first (model-theoretic) approach

Kleene, Spector, Kreisel, Wang, et al.

- $HYP \; = \; \Delta^1_1 \; = \; R_{\omega_1^{CK}} \; = \; L_{\omega_1^{CK}} \cap Pow(\mathbb{N})$.
- Conjecture: Predicatively justifiable subsets of $\mathbb{N} \; = \; HYP$.

However, this approach of iterating predicative set formation involves in an essential way the impredicative notion of being a well-ordering relation, even if one restrictes oneself to recursive well-orderings.

A step away from the semantic notion of well-ordered relation to predicatively provable well-orderings.

The proof-theoretic shift

Solomon Feferman (1928 – 2016)

Kurt Schütte (1909 – 1998)

Feferman – Schütte and the ordinal Γ_0

A boot-strap method

(i) We start off from a predicatively accepted ground theory, say ACA_0.

(ii) Then we systematically extend our framework: Whenever we have proved that a primitive recursive linear ordering is a well-ordering, we are allowed to iterate arithmetic comprehension along this well-ordering and to carry through bar induction along this well-ordering.

Originally done by Feferman and Schütte in the context of systems of ramified analysis or/and progressions of theories.

More modern terminology: the theory $\mathsf{AUT}(\Pi^0_\infty)$

Recall that for any formula $B[n]$ of second order arithmetic,

$$TI[\prec, B] \;:\Leftrightarrow\; Prog[\prec, B] \rightarrow \forall n B[n].$$

$$\mathsf{AUT}(\Pi^0_\infty) \;:=\; \mathsf{ACA}_0 \;+\; \frac{WO[\prec]}{\exists X\, \mathcal{H}_A[\prec, X]} \;+\; (BR)\; \frac{WO[\prec]}{TI[\prec, B]},$$

where $\prec$ is a primitive recursive linear ordering, $A[X, n]$ an arithmetic formula, and $B[n]$ an arbitrary formula.

Theorem

The proof-theoretic ordinal of $\mathsf{AUT}(\Pi^0_\infty)$ *is the ordinal* Γ_0*, and* $L_{\Gamma_0} \cap Pow(\mathbb{N})$ *is its least standard model.*

Reverse Mathematics (Friedman, Simpson, et al.)

Five central subsystems of second order arithmetic – The Big Five

$$\mathsf{RCA}_0 \ - \ \mathsf{WKL}_0 \ - \ \mathsf{ACA}_0 \ - \ \mathsf{ATR}_0 \ - \ \Pi^1_1\text{-}\mathsf{CA}_0$$

The principle (ATR) of arithmetic transfinite recursion

$$\forall R(WO[R] \ \rightarrow \ \exists X\, \mathcal{H}_A[\prec, X]),$$

where $A[X, n]$ is an arbitrary arithmetic formula which may contain additional parameters.

$$\mathsf{ATR}_0 \ := \ \mathsf{ACA}_0 + (\mathsf{ATR})$$

Predicative reducibility of ATR_0

Theorem (Friedman, McAloon, Simpson, J)

1. *The proof-theoretic ordinal of* ATR_0 *is the ordinal* Γ_0.
2. ATR_0 *does not have a minimum* ω*-model or* β*-modell, but HYP is the intersection of all* ω*-models of* ATR_0.
3. Γ_{ε_0} *is the proof-theoretic ordinal of*

$$\mathsf{ATR} \ := \ \mathsf{ATR}_0 + \textit{induction on } \mathbb{N} \textit{ for all } \mathcal{L}_2 \textit{ formulas}$$

First consequences:

(1) $\mathsf{AUT}(\Pi^0_\infty)$ and ATR_0 are proof-theoretically equivalent but conceptually very different.

(2) And is there a big conceptual difference between ATR_0 and ATR?

Equivalences

Fixed points of positive arithmetic clauses (AFP)

$$\exists X \forall n (n \in X \leftrightarrow A[X^+, n]),$$

where $A[X^+, n]$ is an arbitrary X-positive arithmetic formula which may contain additional parameters.

Comparability of well-orderings (CWO)

$$\forall X, Y\, (WO[X] \wedge WO[Y] \rightarrow (|X| \leq |Y| \vee |Y| \leq |X|))$$

Π^1_1 reduction (Π^1_1-Red)

$$\forall n (A[n] \rightarrow B[n]) \rightarrow \exists X (\{n : A[n]\} \subseteq X \subseteq \{n : B[n]\}),$$

where $A[n]$ and $B[n]$ are arbitrary Σ^1_1 and Π^1_1 formulas, respectively.

Theorem (Avigad, Friedman, Simpson)

(ATR), (AFP), (CWO), *and* (Π^1_1-Red) *are pairwise equivalent over* ACA_0.

(Δ^1_1-TR)

$$\forall X \forall n (A[X,n] \leftrightarrow B[X,n]) \land WO[R] \ \rightarrow \ \exists X\, \mathcal{H}_A[R,X]$$

where $A[X,n]$ and $B[X,n]$ are arbitrary Σ^1_1 and Π^1_1 formulas, respectively.

Theorem (Bärtschi, J)

(Δ^1_1-TR) *and* ($\mathrm{M}\Delta^1_1$-FP) *are both equivalent to* (ATR) *over* ACA_0.

From second order arithmetic to set theory

A major difference in building up the universe

- Second order arithmetic: Start off from a fixed/completed ground structure,

$$\mathcal{N} = (\mathbb{N}, \text{prim.rec. functions and relations}).$$

 Subsets of $\mathbb{N}$ are then introduced in a controlled way (predicaively, constructively, ...).

- Set theory: Start off from some basic sets and (sometimes) urelements and build new sets accoring to specific rules. In general there is no a priori bound (super collection) to which all sets belong.

Predicativity in set theory

The Platonic approach

We assume that we have a clear understanding of what an ordinal is and that the constructible universe exists,

$$L \;=\; \bigcup_{\alpha \in On} L_\alpha.$$

Then – in the Feferman-Schütte style – those ordinals can be identified that are "predicatively accessible (justified)". It can be shown (with some effort) that

$$\text{Predicative part of } L \;=\; L_{\Gamma_0}.$$

"Building the universe from below" or "predicatively acceptable closure conditions"

For example:

- Closure under pair, union, product, difference, ...
- Fixed points of positive arithmetic operators with set parameters,

$$\Phi_{\mathfrak{A}} : Pow(\omega) \ni X \longmapsto \{n \in \omega : \mathfrak{A}[S, X^{+}, n]\} \in Pow(\omega).$$

  Then:

  – $\Phi_{\mathfrak{A}}$ has a clear predicative meaning.

  – Define a fixed point of $\Phi_{\mathfrak{A}}$ via a pseudo-hierarchy argument.

  – Stays a fixed point independent of possible new sets.

Basic set theory BS^0

Formulated in the usual language $\mathcal{L}_\in$ of set theory with ω as constant for the first infinite ordinal and relation and function constants for all primitive recursive relations on $\mathbb{N}/\omega$.

Set-theoretic axioms of BS^0

(1) Equality and extensionality,

(2) closure under the rudimentary operations,

(3) Δ_0-Separation: For any Δ_0 formula $A[x]$,

$$\exists y \forall x (x \in y \leftrightarrow x \in a \land A[x]),$$

(4) ω-induction: $(\forall x \subseteq \omega)(x \neq \varnothing \to (\exists m \in x)(\forall n \in x)(m \leq n))$,

(5) The defining axioms for all primitive recursive relations.

BS^0 is clearly justified on predicative grounds. However, situation becomes more complicated if we turn to extensions of BS^0.

Simpson's ATR_0^{set}

$$\mathsf{BS}^0 + (\mathsf{Reg}) + (\mathsf{Count}) + (\mathsf{Beta}),$$

where

- $(\mathsf{Reg}) \;:\Leftrightarrow\; \forall a(a \neq \varnothing \;\rightarrow\; (\exists x \in a)(\forall y \in a)(y \notin x))$.
- $(\mathsf{Count}) \;:\Leftrightarrow\; \forall a(a \text{ is hereditarily countable})$.
- $Wf[a, r] \;\Leftrightarrow\; (\forall b \subseteq a)(b \neq \varnothing \;\rightarrow\; (\exists x \in b)(\forall y \in b)(\langle y, x\rangle \notin r))$,
- $Cp[a, r, f] \;:\Leftrightarrow\; \begin{cases} Dom[f] = a \;\wedge \\ (\forall x \in a)(f(x) = \{f(y) : y \in a \wedge \langle y, x\rangle \in r\}) \end{cases}$
- $(\mathsf{Beta}) \;:\Leftrightarrow\; Wf[a, r] \;\rightarrow\; \exists f\, Cp[a, r, f]$.

Theorem (Simpson)

Every axiom of ATR_0 *is a theorem of* ATR_0^{set} *modulo the natural translation of* $\mathcal{L}_2$ *into* $\mathcal{L}_\in$*.*

Theorem (Simpson)

If A *is an axiom of* ATR_0^{set}*, then* $|A|$ *is a theorem of* ATR_0*.*

$$|A| \;\; ::: \;\; \begin{cases} \text{translation of the } \mathcal{L}_\in \text{ formula } A \text{ into the language } \mathcal{L}_2; \\ \text{sets are represented as well-founded trees;} \\ S \in^* T \;\; :\Leftrightarrow \;\; \exists n(\langle n \rangle \in T \wedge S \simeq T^{\langle n \rangle}) \end{cases}$$

Some aspects of this translation:

- Closure of $\in^*$ under $\simeq$ is required because of extensionality.
- $\in^*$ has a Σ^1_1 definition; with some extra effort it can be made Δ^1_1 in ATR_0.
- The translation of (Beta) is (more or less) for free under this interpretation of $\mathcal{L}_\in$ into $\mathcal{L}_2$.

Question

Is there a natural translation of $\mathcal{L}_\in$ into $\mathcal{L}_2$ that avoids the use of well-founded trees or graphs with specific decorations?

For example, is there a natural interpretation of $\mathcal{L}_\in$ into $\mathcal{L}_2$ – respecting extensionality – that can be developed within Σ^1_1-AC?

Hierarchies, fixed points, and reductions

The obvious analogues of (ATR), (AFP), and (Π^1_1-Red)?

The principle (Δ_0-TR) of Δ_0 transfinite recursion

$$(\forall r \subseteq \omega)(WO[r] \ \rightarrow \ (\exists x \subset \omega)\mathcal{H}_A[r, x])$$

where $A[X, n]$ is an arbitrary Δ_0 which may contain additional parameters.

Fixed points of positive Δ_0 clauses (Δ_0-FP)

$$(\exists x \subseteq \omega)(\forall n \in \omega)(n \in x \ \leftrightarrow \ A[x^+, n]),$$

where $A[x^+, n]$ is an arbitrary x-positive arithmetic formula which may contain additional parameters.

Π reduction (Π-Red)

$$(\forall x \in a)(A[x] \to B[x]) \;\to\; \exists y(\{x \in a : A[x]\} \subseteq y \subseteq \{x \in a : B[x]\}),$$

where $A[x]$ and $B[x]$ are arbitrary Σ and Π formulas, respectively.

Theorem (Bärtschi, J)

1. $\mathsf{BS}^0 + (\Delta_0\text{-FP}) \;\vdash\; (\Delta_0\text{-TR})$.
2. $\mathsf{BS}^0 + (\Pi\text{-Red}) \;\vdash\; (\Delta_0\text{-TR})$.
3. $\mathsf{ATR}_0 \;\subseteq\; \mathsf{BS}^0 + (\Delta_0\text{-TR}) \;\subseteq\; \begin{cases} \mathsf{BS}^0 + (\Delta_0\text{-FP}), \\ \mathsf{BS}^0 + (\Pi\text{-Red}). \end{cases}$

Theorem

1. $\mathsf{BS}^0 + (\Pi\text{-Red}) \ \leq \ \mathsf{ATR}_0$.
2. *The proof-theoretic ordinal of* $\mathsf{BS}^0 + (\Delta_0\text{-FP})$ *is* Γ_0.

The first reduction is via a modified Simpson translation of $\mathcal{L}_\in$ into $\mathcal{L}_2$, the second via an embedding into KPi^0.

Question

What is the exact relationship – over BS^0 – between

$$(\Delta_0\text{-TR}), \quad (\Delta_0\text{-FP}), \quad (\Pi\text{-Red})?$$

Kripke-Platek set theory KP

KP $:=$ BS^0 plus the following two axiom schemes

(1) (Δ_0-*Collection*): For all Δ_0 formulas $A[x,y]$,

$$(\forall x \in a)\exists y A[x,y] \;\rightarrow\; \exists z(\forall x \in a)(\exists y \in z)A[x,y].$$

(2) ($\mathcal{L}_\in$-$\mathsf{I}_\in$): For all $\mathcal{L}_\in$ formulas$B[x]$,

$$\forall x((\forall y \in x)B[y] \rightarrow B[x]) \;\rightarrow\; \forall x B[x].$$

Relationship between ATR_0^{set} and KP

Theorem (J)

1. *The proof-theoretic ordinal of* KP *is the Bachmann-Howerd ordinal;* KP *is proof-theoretically equivalent to the theory* ID_1.
2. $\mathsf{KP} + (\mathsf{Beta})$ *is proof-theoretically equivalent to* $\Delta^1_2\text{-}\mathsf{CA} + (\mathsf{BI})$.

Immediate consequence

$$\mathsf{KP} \not\subseteq \mathsf{ATR}_0^{set} \quad \text{and} \quad \mathsf{ATR}_0^{set} \not\subseteq \mathsf{KP}.$$

Further:

- KP $\nvdash$ $(\mathrm{AFP})^-$ (Gregoriades for parameter-free).
- KP + (AFP) and KP have the same proof-theoretic strength (Sato).
- KP + (Beta) $\vdash$ (Δ_0-FP).
- KP + (Beta) + (Π-Red) proves Π^1_2 comprehension on ω.
- KP + ($V{=}L$) + (Π-Red) proves Π^1_2 comprehension on ω.

Question

But what can we say about KP + (Π-Red)?

Kripke-Platek without foundation and extensions

$$\mathsf{KP}^0 \;:=\; \mathsf{BS}^0 + (\Delta_0\text{-}\mathit{Collection})$$

$\mathcal{L}_\in \;:=\; \mathcal{L}_\in(\mathsf{Ad})$ with Ad a unary relation symbol to express admissibility.

Ad axioms

(Ad.1) $\mathsf{Ad}(d) \;\to\; d \text{ transitive} \wedge \omega \in d.$

(Ad.2) $\mathsf{Ad}(d) \;\to\; A^d$ for every closed instance of an axiom of KP^0.

(Ad.3) $\mathsf{Ad}(d_1) \wedge \mathsf{Ad}(d_2) \;\to\; d_1 \in d_2 \vee d_1 = d_2 \vee d_2 \in d_1.$

$$\mathsf{KPi}^0 \;:=\; \mathsf{KP}^0 + \forall x \exists y (x \in y \wedge \mathsf{Ad}(y)),$$

$$\mathsf{KPi} \;:=\; \mathsf{KP} + \forall x \exists y (x \in y \wedge \mathsf{Ad}(y)).$$

Remark

1. The least α such that $L_\alpha \models \mathsf{KPi}$ is the first rec. inacc. ordinal.
2. KPi^0 poves (Beta). However, (Beta) is very weak in the context of KPi^0 since there is no induction on the ordinals.
3. On the other hand, it is strong in KP since then it makes the Π_1 predicate "r is well-founded on a" a Δ_1 predicate.

Theorem (J)

1. $\mathsf{ATR}_0 \subseteq \mathsf{KPi}^0$ *and the proof-theoretic ordinal of* KPi^0 *is* Γ_0.
2. KPi *is proof-theoretically equivalent to* $\mathsf{KP} + (\mathsf{Beta})$, *and thus also to* $\Delta^1_2\text{-}\mathsf{CA} + (\mathsf{BI})$.

Outlook

- The relationship between subsystems of second order arithmetic and set theory is rather transparent *as soon as Axiom (Beta) is available*.
- However, what can we say if we do not have Axiom (Beta)? Is there a general picture?
- Is Axiom (Beta) a philosophically relevant principle?
- The *fat versus high* question.

Thank you for your attention!