Parameter-Free Versions of ATR_0 and Related Theories

Michael Bärtschi
jww Gerhard Jäger
Universität Bern

ABM Meeting Spring 2019
Ludwig-Maximilians-Universität München
May 2 & 3

Outline

General context

We are working in second order arithmetic, i.e., our language $\mathcal{L}_2$ is two-sorted featuring number and set variables. In general our base theory will be

$$\mathsf{ACA}_0$$

featuring the induction axiom

$$(0 \in X \wedge \forall n(n \in X \rightarrow n+1 \in X)) \rightarrow \forall n(n \in X)$$

and arithmetic comprehension, i.e.,

$$\exists X \forall n(n \in X \leftrightarrow A(n))$$

for any arithmetic formula A.

$$\mathsf{ACA} = \mathsf{ACA}_0 + \text{full second order induction scheme}$$

Linear orderings

We fix some pairing map, e.g.,

$$(m, n) := \frac{1}{2}(m+n)(m+n+1)+m$$

A set $R \subseteq \mathbb{N} \times \mathbb{N}$ is reflexive if

$$(m, n) \in R \text{ implies } (m, m), (n, n) \in R$$

Linear orderings

We fix some pairing map, e.g.,

$$(m, n) := \frac{1}{2}(m+n)(m+n+1) + m$$

A set $R \subseteq \mathbb{N} \times \mathbb{N}$ is reflexive if

$$(m, n) \in R \text{ implies } (m, m), (n, n) \in R$$

For reflexive R we define:

$$\begin{aligned}
\text{field}(R) &:= \{m \colon (m, m) \in R\} \\
m \leq_R n &:= (m, n) \in R \\
m <_R n &:= (m, n) \in R \wedge (n, m) \notin R \\
\text{LO}(R) &:= R \text{ is a linear order} \\
\text{WO}(R) &:= R \text{ has no infinite descending chain}
\end{aligned}$$

Linear orderings

We fix some pairing map, e.g.,

$$(m,n) := \tfrac{1}{2}(m+n)(m+n+1)+m$$

A set $R \subseteq \mathbb{N}\times\mathbb{N}$ is reflexive if

$$(m,n)\in R \text{ implies } (m,m),(n,n)\in R$$

For reflexive R we define:

$$\begin{aligned}
\text{field}(R) &:= \{m\colon (m,m)\in R\}\\
m \leq_R n &:= (m,n)\in R\\
m <_R n &:= (m,n)\in R \wedge (n,m)\notin R\\
\text{LO}(R) &:= R \text{ is a linear order}\\
\text{WO}(R) &:= R \text{ has no infinite descending chain}
\end{aligned}$$

We fix a primitive recursive well-ordering $\lhd$ up to Γ_0. For $n \in \text{field}(\lhd)$ set:

$$\lhd \restriction n := \{m\colon m \lhd n\}$$

Hierarchies

For any $\mathrm{LO}(R)$, set $X \subseteq \mathbb{N} \times \mathrm{field}(R)$ and $j \in \mathrm{field}(R)$ we define

$$X_j := \{n\colon (n,j) \in X\}$$
$$X^{Rj} := \{(n,i)\colon i <_R j \wedge (n,i) \in X\}$$

Intuitively $X^{Rj} = \bigoplus_{i <_R j} X_i$. Define

Hierarchies

For any $\mathrm{LO}(R)$, set $X \subseteq \mathbb{N} \times \mathrm{field}(R)$ and $j \in \mathrm{field}(R)$ we define

$$X_j := \{n \colon (n,j) \in X\}$$
$$X^{Rj} := \{(n,i) \colon i <_R j \wedge (n,i) \in X\}$$

Intuitively $X^{Rj} = \bigoplus_{i <_R j} X_i$. Define

$$H_A(R,X) := LO(R) \wedge X = \big\{(n,j) \colon j \in \mathrm{field}(R) \wedge A(n,j,X^{Rj})\big\}$$

for a formula $A(n,j,X)$. So for $j \in \mathrm{field}(R)$ we have

$$X_j = \big\{n \colon A(n,j,X^{Rj})\big\}$$

Axiom schemas

For a formula $B(\vec{x}, \vec{X})$, free variables different from the ones indicated, i.e., $\vec{x}, \vec{X}$, are called parameters. We consider

$$\mathfrak{A} := \left\{ \; A(\vec{x}, \vec{X}) \colon A \text{ arithmetic, with } \textbf{set} \text{ and } \textbf{number} \text{ parameters} \right\}$$

$$\mathfrak{A}^- := \left\{ A^-(\vec{x}, \vec{X}) \colon A^- \text{ arithmetic, with } \textbf{number} \text{ parameters } \textbf{only} \right\}$$

Axiom schemas

For a formula $B(\vec{x}, \vec{X})$, free variables different from the ones indicated, i.e., $\vec{x}, \vec{X}$, are called parameters. We consider

$$\mathfrak{A} := \left\{ \; A(\vec{x}, \vec{X}) \colon A \text{ arithmetic, with } \textbf{set} \text{ and } \textbf{number} \text{ parameters} \right\}$$

$$\mathfrak{A}^- := \left\{ A^-(\vec{x}, \vec{X}) \colon A^- \text{ arithmetic, with } \textbf{number} \text{ parameters } \textbf{only} \right\}$$

(Variants of) arithmetic transfinite recursion:

$$\forall A(n, j, X) \in \mathfrak{A} \; : \; \forall R(\mathrm{WO}(R) \to \exists X\, H_A(R, X)) \qquad (\mathsf{ATR})$$

$$\forall A^-(n, j, X) \in \mathfrak{A}^- : \; \forall R(\mathrm{WO}(R) \to \exists X\, H_{A^-}(R, X)) \qquad (\mathsf{ATR}^-)$$

$$\forall A^-(n, j, X) \in \mathfrak{A}^- : \; \forall m(\mathrm{WO}(\lhd \restriction m) \to \exists X\, H_{A^-}(\lhd \restriction m, X)) \qquad (\mathsf{prATR}^-)$$

Axiom schemas

For a formula $B(\vec{x}, \vec{X})$, free variables different from the ones indicated, i.e., $\vec{x}, \vec{X}$, are called parameters. We consider

$$\mathfrak{A} := \left\{ \ A(\vec{x}, \vec{X}) \colon A \text{ arithmetic, with } \textbf{set} \text{ and } \textbf{number} \text{ parameters} \right\}$$

$$\mathfrak{A}^- := \left\{ A^-(\vec{x}, \vec{X}) \colon A^- \text{ arithmetic, with } \textbf{number} \text{ parameters } \textbf{only} \right\}$$

(Variants of) arithmetic transfinite recursion:

$$\forall A(n,j,X) \in \mathfrak{A} : \quad \forall R(\mathrm{WO}(R) \to \exists X\, H_A(R,X)) \qquad (\mathsf{ATR})$$

$$\forall A^-(n,j,X) \in \mathfrak{A}^- : \quad \forall R(\mathrm{WO}(R) \to \exists X\, H_{A^-}(R,X)) \qquad (\mathsf{ATR}^-)$$

$$\forall A^-(n,j,X) \in \mathfrak{A}^- : \forall m(\mathrm{WO}(\lhd \restriction m) \to \exists X\, H_{A^-}(\lhd \restriction m, X)) \qquad (\mathsf{prATR}^-)$$

(Variants of) Arithmetic fixed points:

$$\forall A(n, Y^+) \in \mathfrak{A} : \exists Y\, \forall n\, (n \in Y \leftrightarrow A(n, Y^+)) \qquad (\mathsf{FP})$$

$$\forall A^-(n, Y^+) \in \mathfrak{A}^- : \exists Y\, \forall n\, (n \in Y \leftrightarrow A^-(n, Y^+)) \qquad (\mathsf{FP}^-)$$

where Y occurs only positively in A, A^-.

Formal systems

$$\begin{aligned}
\mathsf{ATR}_{(0)} &:= \mathsf{ACA}_{(0)} + (\mathsf{ATR}) \\
\mathsf{ATR}^{-}_{(0)} &:= \mathsf{ACA}_{(0)} + (\mathsf{ATR}^{-}) \\
\mathsf{prATR}^{-}_{(0)} &:= \mathsf{ACA}_{(0)} + (\mathsf{prATR}^{-}) \\
\mathsf{FP}_{(0)} &:= \mathsf{ACA}_{(0)} + (\mathsf{FP}) \\
\mathsf{FP}^{-}_{(0)} &:= \mathsf{ACA}_{(0)} + (\mathsf{FP}^{-})
\end{aligned}$$

Formal systems

$$
\begin{aligned}
\mathsf{ATR}_{(0)} &:= \mathsf{ACA}_{(0)} + (\mathsf{ATR}) \\
\mathsf{ATR}^{-}_{(0)} &:= \mathsf{ACA}_{(0)} + (\mathsf{ATR}^{-}) \\
\mathsf{prATR}^{-}_{(0)} &:= \mathsf{ACA}_{(0)} + (\mathsf{prATR}^{-}) \\
\mathsf{FP}_{(0)} &:= \mathsf{ACA}_{(0)} + (\mathsf{FP}) \\
\mathsf{FP}^{-}_{(0)} &:= \mathsf{ACA}_{(0)} + (\mathsf{FP}^{-})
\end{aligned}
$$

Goal: Characterise the set-parameter free systems up to proof-theoretic strength.

First observations

Lemma 1

ATR_0^- *proves all instances of* (ATR)

Proof sketch: Let R be a well-ordering and Y a set. We define a well-ordering S such that

$$\begin{aligned} \text{field}(S) &= (\{0\} \times Y) \cup \{(1,1)\} \cup (\{2\} \times R) \\ &= \text{disjoint sum of } Y,\ \{1\}\ \&\ R \end{aligned}$$

First observations

Lemma 1

ATR_0^- *proves all instances of* (ATR)

Proof sketch: Let R be a well-ordering and Y a set. We define a well-ordering S such that

$$\begin{aligned} \text{field}(S) &= (\{0\} \times Y) \cup \{(1,1)\} \cup (\{2\} \times R) \\ &= \text{disjoint sum of } Y,\ \{1\}\ \&\ R \end{aligned}$$

where the ordering S looks like

$$(Y, <_{\mathbb{N}}) <_S (1,1) <_S (R, <_R)$$

S is a well-ordering, also Y and R can be obtained from S by arithmetic comprehension. $(1,1)$ serves as a separator.

Next consider $A(n, j, X, Y)$ arithmetic, with only X, Y as set variables. The goal is to specify a transformation

$$A(n, j, X, Y) \quad \overset{into}{\leadsto} \quad B(n, j, Z)$$

apply (ATR^{-}) to B, yielding a hierarchy W along S iterating B, i.e.,

$$W_{(k,j)} = \left\{n \colon B(n, (k, j), W^{S(k.j)})\right\} \quad \text{for } (k, j) \in \text{field}(S)$$

such that

Next consider $A(n, j, X, Y)$ arithmetic, with only X, Y as set variables. The goal is to specify a transformation

$$A(n, j, X, Y) \quad \overset{into}{\rightsquigarrow} \quad B(n, j, Z)$$

apply (ATR^-) to B, yielding a hierarchy W along S iterating B, i.e.,

$$W_{(k,j)} = \left\{ n\colon B(n, (k, j), W^{S(k.j)}) \right\} \quad \text{for } (k, j) \in \text{field}(S)$$

such that

- if $k = 0$, W copies Y, one element per level
- if $k = 1$, W uses $(1, 1)$ as separator
- if $k = 2$, W iterates A along R. X and Y are encoded in Z.

Taking the levels of W above $(1, 1)$ gives the desired hierarchy for A. □

Next consider $A(n, j, X, Y)$ arithmetic, with only X, Y as set variables. The goal is to specify a transformation

$$A(n, j, X, Y) \quad \overset{into}{\rightsquigarrow} \quad B(n, j, Z)$$

apply (ATR^-) to B, yielding a hierarchy W along S iterating B, i.e.,

$$W_{(k,j)} = \left\{ n \colon B(n, (k, j), W^{S(k.j)}) \right\} \quad \text{for } (k, j) \in \text{field}(S)$$

such that

- if $k = 0$, W copies Y, one element per level
- if $k = 1$, W uses $(1, 1)$ as separator
- if $k = 2$, W iterates A along R. X and Y are encoded in Z.

Taking the levels of W above $(1, 1)$ gives the desired hierarchy for A. □

Corollary 1

$$\text{ATR}_0^- \approx \text{ATR}_0 \ \text{ATR}^- \approx \text{ATR}$$

Avigad (1996) showed the equivalence of

$$\text{(ATR) and (FP) over ACA}_0$$

The next two lemmas follow in essentially the same manner.

Avigad (1996) showed the equivalence of

$$\text{(ATR) and (FP) over } \mathsf{ACA}_0$$

The next two lemmas follow in essentially the same manner.

Lemma 2 (following Avigad (1996))

$\mathsf{FP}_0^- \vdash (\mathsf{prATR}^-)$

Proof sketch: Working in FP_0^- assume $\mathrm{WO}(\lhd \restriction m)$ and let

$$A^-(n, j, X) \quad \text{arithmetical, no set parameters, in nnf}$$

Idea: Derive the char. function of the desired hierarchy by transforming

$$A^-(n, j, X) \quad \overset{into}{\rightsquigarrow} \quad A^+(n, j, Z^+, \lhd, b)$$

Avigad (1996) showed the equivalence of

$$\text{(ATR) and (FP) over ACA}_0$$

The next two lemmas follow in essentially the same manner.

Lemma 2 (following Avigad (1996))

$\mathsf{FP}_0^- \vdash (\mathsf{prATR}^-)$

Proof sketch: Working in FP_0^- assume $\mathrm{WO}(\lhd \restriction m)$ and let

$$A^-(n,j,X) \quad \text{arithmetical, no set parameters, in nnf}$$

Idea: Derive the char. function of the desired hierarchy by transforming

$$A^-(n,j,X) \quad \overset{into}{\rightsquigarrow} \quad A^+(n,j,Z^+,\lhd,b)$$

by replacing in A^-

- $t \in X$ with $(\,(t,1) \in Z \wedge \quad (t_0 \lhd b)\,)$
- $t \notin X$ with $(\,(t,0) \in Z \vee \neg(t_0 \lhd b)\,)$

Analogously we transform $\neg(A^-) \overset{into}{\rightsquigarrow} (\neg A)^+$.

Using A^+, $(\neg A)^+$ we define a Z-positive arithmetical formula

$$B\left(\left(\,(n,b),k\,\right),j,Z^+,\lhd\right)$$

which asserts the following:

Using A^+, $(\neg A)^+$ we define a Z-positive arithmetical formula

$$B\left(\left(\,(n,b),k\,\right),j,Z^+,\lhd\right)$$

which asserts the following:

- For any $l <_R b$ and n:

$$\left(\,(n,l)\,,\,0\,\right)\in Z \quad \text{or} \quad \left(\,(n,l)\,,\,1\,\right)\in Z$$

- At level b:

$$\left(\,(n,b)\,,\,1\,\right),\left(\,(n,b)\,,\,0\,\right)\in Z \quad \text{according to } A^+,(\neg A)^+$$

Using A^+, $(\neg A)^+$ we define a Z-positive arithmetical formula

$$B\left(\,(\,(n,b),k\,)\,,j,Z^+,\lhd\right)$$

which asserts the following:

- For any $l <_R b$ and n:

$$(\,(n,l)\,,\,0\,) \in Z \quad \text{or} \quad (\,(n,l)\,,\,1\,) \in Z$$

- At level b:

$$(\,(n,b)\,,\,1\,)\,,\,(\,(n,b)\,,\,0\,) \in Z \quad \text{according to } A^+, (\neg A)^+$$

(FP^-) is applicable, yielding a fixed point Y:

$$(\,(n,b)\,,\,k\,) \in Y \leftrightarrow B\left(\,(\,(n,b)\,,\,k\,)\,,j,Y^+,\lhd\,\right)$$

Since $\mathrm{WO}(\lhd \restriction m)$, Y actually represents a hierarchy X. By construction X iterates A^- along $\lhd \restriction m$. □

Using A^+, $(\neg A)^+$ we define a Z-positive arithmetical formula

$$B\left(\left(\,(n,b),k\,\right),j,Z^+,\lhd\right)$$

which asserts the following:

- For any $l <_R b$ and n:

$$(\,(n,l)\,,\,0\,) \in Z \quad \text{or} \quad (\,(n,l)\,,\,1\,) \in Z$$

- At level b:

$$(\,(n,b)\,,\,1\,)\,,\,(\,(n,b)\,,\,0\,) \in Z \quad \text{according to } A^+,(\neg A)^+$$

(FP^-) is applicable, yielding a fixed point Y:

$$(\,(n,b)\,,\,k\,) \in Y \leftrightarrow B\left(\left(\,(n,b)\,,\,k\,\right),j,Y^+,\lhd\right)$$

Since $\mathrm{WO}(\lhd \restriction m)$, Y actually represents a hierarchy X. By construction X iterates A^- along $\lhd \restriction m$. □

Corollary 2

$$\mathsf{prATR}_0^- \subseteq \mathsf{FP}_0^- \quad \& \quad \mathsf{prATR}^- \subseteq \mathsf{FP}^-$$

Lemma 3 (following Avigad (1996))

$\mathsf{ATR}_0^- \vdash (\mathsf{FP}^-)$

Proof sketch: The proof involves pseudohierarchies (Spector, 1959; Gandy, 1960). By Corollary 1 we can work in ATR_0. Let $A^-(n, Y^+)$ be arithmetic with no set parameters. We stipulate

$$B(R) \equiv \mathrm{LO}(R) \wedge \exists x(x = \min(R)) \wedge \exists X(X \text{ hierarchy along } R \text{ satisfiying } ❶ - ❹),$$

where for all $j, k \in \mathrm{field}(R)$

Lemma 3 (following Avigad (1996))

$\mathsf{ATR}_0^- \vdash (\mathsf{FP}^-)$

Proof sketch: The proof involves pseudohierarchies (Spector, 1959; Gandy, 1960). By Corollary 1 we can work in ATR_0. Let $A^-(n, Y^+)$ be arithmetic with no set parameters. We stipulate

$$B(R) \equiv \mathrm{LO}(R) \wedge \exists x(x = \min(R)) \wedge \\ \exists X(X \text{ hierarchy along } R \text{ satisfiying } ❶ - ❹),$$

where for all $j, k \in \mathrm{field}(R)$

1. $X_0 = \emptyset \wedge (\, j \text{ is a successor} \rightarrow X_{j+1} = \{n \colon A^-(n, X_j)\}\,)$
2. j is a limit $\rightarrow X_j = \bigcup_{i <_R j} X_i$
3. $n \in X_j \rightarrow$ n entered the hierarchy at some level
4. $j <_R k \rightarrow X_j \subseteq X_k$

We can show that

$$\mathsf{ATR}_0 \vdash \mathrm{WO}(R) \to B(R)$$

For ❶, ❷, ❸ we need R and j. ❹ relies on the Y-positivity of A^-. $B(R)$ is Σ^1_1, hence by the Π^1_1-universality of $\mathrm{WO}(R)$, i.e.,

Theorem (Π^1_1-universality of $\mathrm{WO}(R)$)

For any Σ^1_1 formula $C(X)$, ACA_0 proves

$$\exists W\,(\,\mathrm{WO}(W) \nleftrightarrow C(W)\,).$$

We can show that

$$\mathsf{ATR}_0 \vdash \mathrm{WO}(R) \to B(R)$$

For ❶, ❷, ❸ we need R and j. ❹ relies on the Y-positivity of A^-. $B(R)$ is Σ^1_1, hence by the Π^1_1-universality of $\mathrm{WO}(R)$, i.e.,

Theorem (Π^1_1-universality of $\mathrm{WO}(R)$)

For any Σ^1_1 formula $C(X)$, ACA_0 proves

$$\exists W\,(\,\mathrm{WO}(W) \nleftrightarrow C(W)\,).$$

we get

$$\mathsf{ATR}_0 \vdash \exists W\,\underbrace{(\,B(W) \nrightarrow \mathrm{WO}(W)\,)}_{B(W)\wedge\neg\mathrm{WO}(W)}.$$

W is a so-called pseudohierarchy. A fixed point for $A^-(n, Y^+)$ can be obtained from W by using ❶ - ❹ and arithmetic comprehension. □

We can show that

$$\mathsf{ATR}_0 \vdash \mathrm{WO}(R) \to B(R)$$

For ❶, ❷, ❸ we need R and j. ❹ relies on the Y-positivity of A^-. $B(R)$ is Σ^1_1, hence by the Π^1_1-universality of $\mathrm{WO}(R)$, i.e.,

Theorem (Π^1_1-universality of $\mathrm{WO}(R)$)

For any Σ^1_1 formula $C(X)$, ACA_0 proves

$$\exists W\,(\,\mathrm{WO}(W) \leftrightarrow C(W)\,).$$

we get

$$\mathsf{ATR}_0 \vdash \exists W\ \underbrace{(\,B(W) \not\to \mathrm{WO}(W)\,)}_{B(W)\wedge\neg\mathrm{WO}(W)}.$$

W is a so-called pseudohierarchy. A fixed point for $A^-(n, Y^+)$ can be obtained from W by using ❶ - ❹ and arithmetic comprehension. □

Corollary 3

$$\mathsf{FP}_0^- \subseteq \mathsf{ATR}_0^- \qquad \mathsf{FP}^- \subseteq \mathsf{ATR}^-$$

Well-ordering proofs

We use the binary Veblen functions $\varphi_\alpha(\beta)$, i.e. $\varphi_0(\alpha) = \omega^\alpha$ and

$$\varphi_\alpha \colon \mathrm{Ord} \to \mathrm{Ord}$$

enumerates the common fixed points of all φ_β with $\beta < \alpha$. Γ_0 denotes the first ordinal such that $\varphi_{\Gamma_0}(0) = \Gamma_0$.

Well-ordering proofs

We use the binary Veblen functions $\varphi_\alpha(\beta)$, i.e. $\varphi_0(\alpha) = \omega^\alpha$ and

$$\varphi_\alpha \colon \mathrm{Ord} \to \mathrm{Ord}$$

enumerates the common fixed points of all φ_β with $\beta < \alpha$. Γ_0 denotes the first ordinal such that $\varphi_{\Gamma_0}(0) = \Gamma_0$. Using $\lhd$ we establish upper proof-theoretic bounds relative to an anonymous relation variable Q.

Abbreviations:

$$\begin{aligned} \alpha \subseteq S &:= (\forall \xi \lhd \alpha)(\xi \in S) \\ \mathrm{Prog}(S) &:= \forall \alpha(\alpha \subseteq S \to \alpha \in S) \\ \mathrm{TI}(S,\alpha) &:= \mathrm{Prog}(S) \to \alpha \subseteq S \end{aligned}$$

Well-ordering proofs

We use the binary Veblen functions $\varphi_\alpha(\beta)$, i.e. $\varphi_0(\alpha) = \omega^\alpha$ and

$$\varphi_\alpha \colon \mathrm{Ord} \to \mathrm{Ord}$$

enumerates the common fixed points of all φ_β with $\beta < \alpha$. Γ_0 denotes the first ordinal such that $\varphi_{\Gamma_0}(0) = \Gamma_0$. Using $\lhd$ we establish upper proof-theoretic bounds relative to an anonymous relation variable Q.

Abbreviations:

$$\begin{aligned}
\alpha \subseteq S &:= (\forall \xi \lhd \alpha)(\xi \in S) \\
\mathrm{Prog}(S) &:= \forall \alpha(\alpha \subseteq S \to \alpha \in S) \\
\mathrm{TI}(S, \alpha) &:= \mathrm{Prog}(S) \to \alpha \subseteq S \\
\mathrm{Sp}(S) &:= \lambda \alpha. \forall \xi(\xi \subseteq S \to \xi + \alpha \subseteq S) \\
\mathcal{W}(\alpha) &:= \mathrm{TI}(Q, \alpha)
\end{aligned}$$

We set

$$\widehat{\varphi}(\alpha, \eta) := \varphi_{\alpha_1}(\varphi_{\alpha_2}(\dots(\varphi_{\alpha_n}(\varphi_{\mathrm{e}(\alpha)}(\eta)) \dots)$$

where

$$\alpha \overset{\mathsf{NF}}{=} \underbrace{\omega^{\alpha_1} + \dots + \omega^{\alpha_n}}_{\mathrm{h}(\alpha)} + \omega^{e(\alpha)} \quad \text{and} \quad \alpha_1 \trianglerighteq \dots \trianglerighteq \alpha_n \trianglerighteq \mathrm{e}(\alpha)\,,$$

We set

$$\widehat{\varphi}(\alpha, \eta) := \varphi_{\alpha_1}(\varphi_{\alpha_2}(\ldots(\varphi_{\alpha_n}(\varphi_{\mathrm{e}(\alpha)}(\eta))\ldots)$$

where

$$\alpha \overset{\mathsf{NF}}{=} \underbrace{\omega^{\alpha_1} + \ldots + \omega^{\alpha_n}}_{\mathrm{h}(\alpha)} + \omega^{e(\alpha)} \quad \text{and} \quad \alpha_1 \trianglerighteq \ldots \trianglerighteq \alpha_n \trianglerighteq \mathrm{e}(\alpha)\,,$$

and define the jump operator

$$\mathrm{Sp}^*(S, \alpha) := \lambda\beta.(\forall \xi \lhd \alpha)(\,\varphi_{e(\alpha)}(\beta) \in \mathrm{Sp}(S_\xi)\,).$$

Finally we consider hierarchies

$$\mathcal{R}(S, \tilde{\alpha}) := (\forall \alpha \trianglelefteq \tilde{\alpha})(\,0 \lhd \alpha \to S_\alpha = \mathrm{Sp}^*(S, \alpha)\,)$$

Lemma 4 (Lower proof-theoretic bounds for $\mathsf{prATR}^-_{(0)}$)

$$\mathsf{prATR}^-_0 \vdash \mathcal{W}(\alpha) \textit{ for all } \alpha \lhd \varphi_{\varepsilon_0}(0)$$

$$\mathsf{prATR}^- \vdash \mathcal{W}(\alpha) \textit{ for all } \alpha \lhd \varphi_{\varphi_1(\varepsilon_0)}(0)$$

Proof sketch: We use the following theorem adapted to our setting featuring the anonymous relation Q:

Lemma 4 (Lower proof-theoretic bounds for $\mathsf{prATR}^-_{(0)}$)

$$\mathsf{prATR}^-_0 \vdash \mathcal{W}(\alpha) \text{ for all } \alpha \lhd \varphi_{\varepsilon_0}(0)$$

$$\mathsf{prATR}^- \vdash \mathcal{W}(\alpha) \text{ for all } \alpha \lhd \varphi_{\varphi_1(\varepsilon_0)}(0)$$

Proof sketch: We use the following theorem adapted to our setting featuring the anonymous relation Q:

Theorem

$\mathsf{ACA}_0 + \{\mathrm{TI}(A, \alpha) \colon A(x) \text{ arithmetic}\}$ *proves that*

$$R(Y, \alpha) \wedge \beta \subseteq Y_\alpha \to \mathrm{TI}(Y_0, \widehat{\varphi}(\alpha, \beta))$$

Lemma 4 (Lower proof-theoretic bounds for $\mathsf{prATR}^-_{(0)}$)

$$\mathsf{prATR}_0^- \vdash \mathcal{W}(\alpha) \textit{ for all } \alpha \lhd \varphi_{\varepsilon_0}(0)$$

$$\mathsf{prATR}^- \vdash \mathcal{W}(\alpha) \textit{ for all } \alpha \lhd \varphi_{\varphi_1(\varepsilon_0)}(0)$$

Proof sketch: We use the following theorem adapted to our setting featuring the anonymous relation Q:

Theorem

$\mathsf{ACA}_0 + \{\mathrm{TI}(A, \alpha)\colon A(x) \textit{ arithmetic}\}$ *proves that*

$$R(Y, \alpha) \wedge \beta \subseteq Y_\alpha \to \mathrm{TI}(Y_0, \widehat{\varphi}(\alpha, \beta))$$

Using (prATR^-) there exists a hierarchy Y along $\lhd \restriction m$ such that $Y_0 = Q$, $\mathcal{R}(Y, m)$. Then we use $0 \subseteq Q$ and that the fact that for arithmetic $A(x)$

$$\mathsf{ACA}_0 \vdash \mathrm{TI}(A(x), \alpha) \text{ for all } \alpha \lhd \varepsilon_0$$

$$\mathsf{ACA} \vdash \mathrm{TI}(A(x), \alpha) \text{ for all } \alpha \lhd \varphi_1(\varepsilon_0)$$

Corollary 4

$|\mathrm{prATR}_0^-| \geq \varphi_{\varepsilon_0}(0) \quad \& \quad |\mathrm{prATR}^-| \geq \varphi_{\varphi_1(\varepsilon_0)}(0)$

By showing that FP_0^- is conservative over $|\widehat{\mathrm{ID}}_1|$ it follows that

Lemma 5

$|\mathrm{FP}_0^-| = |\widehat{\mathrm{ID}}_1| = \varphi_{\varepsilon_0}(0)$

Combining everything so far gives

$$\begin{aligned} \mathrm{ATR}_0^- &\approx \mathrm{ATR}_0 \\ |\mathrm{prATR}_0^-| &= |\mathrm{FP}_0^-| = \varphi_{\varepsilon_0}(0) \\ \mathrm{ATR}^- &\approx \mathrm{ATR} \\ \mathrm{prATR}^- &\subseteq \mathrm{FP}^- \\ |\mathrm{prATR}^-| &\geq \varphi_{\varphi_1(\varepsilon_0)}(0) \end{aligned}$$

Corollary 4

$|\text{prATR}_0^-| \geq \varphi_{\varepsilon_0}(0) \quad \& \quad |\text{prATR}^-| \geq \varphi_{\varphi_1(\varepsilon_0)}(0)$

By showing that FP_0^- is conservative over $|\widehat{\text{ID}}_1|$ it follows that

Lemma 5

$|\text{FP}_0^-| = |\widehat{\text{ID}}_1| = \varphi_{\varepsilon_0}(0)$

Combining everything so far gives

$$\begin{aligned} \text{ATR}_0^- &\approx \text{ATR}_0 \\ |\text{prATR}_0^-| = |\text{FP}_0^-| &= \varphi_{\varepsilon_0}(0) \\ \text{ATR}^- &\approx \text{ATR} \\ \text{prATR}^- &\subseteq \text{FP}^- \\ |\text{prATR}^-| &\geq \varphi_{\varphi_1(\varepsilon_0)}(0) \end{aligned}$$

If we can show that $|\text{FP}^-| \leq \varphi_{\varphi_1(\varepsilon_0)}(0)$ it follows that

$$|\text{prATR}^-| = |\text{FP}^-| = \varphi_{\varphi_1(\varepsilon_0)}(0)$$

Thank you for your attention!